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Multiple solutions for the p-Laplace operator with critical growth[☆]

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ABSTRACT

In this paper we show the existence of at least three nontrivial solutions to the following quasilinear elliptic equation $-\Delta_p u = |u|^{p^*-2}u + \lambda f(x,u)$ in a smooth bounded domain Ω of $\mathbb{R}^N$ with homogeneous Dirichlet boundary conditions on $\partial\Omega$, where $p^* = Np/(N-p)$ is the critical Sobolev exponent and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. The proof is based on variational arguments and the classical concentration compactness method.

1. Introduction

Let us consider the following nonlinear elliptic problem:

$$\begin{cases} -\Delta_p u = |u|^{p^*-2}u + \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $1 < p < N$, $p^* = Np/(N-p)$ is the critical exponent in the Sobolev embedding and λ is a positive parameter.

Problems like (P) appear naturally in several branches of pure and applied mathematics, such as the theory of quasiregular and quasiconformal mappings in Riemannian manifolds (see [1,2]), non-Newtonian fluids, reaction–diffusion problems, flow through porous media, nonlinear elasticity, glaciology, etc. (see [3–6]).

The purpose of this paper, is to prove the existence of at least three nontrivial solutions for (P) under adequate assumptions on the source term f and the parameter λ. This result extends an old paper by Struwe [7]. A related result for the nonlinear boundary condition case can be found in [8].

Here, no oddness condition is imposed in f and a positive, a negative and a sign-changing solution are found. (For odd nonlinearities it is well known that infinitely many solutions can be obtained in many situations, by using the tools of critical point theory, as the $\mathbb{Z}_2$-symmetric version of the mountain pass theorem. See for instance [9] or [10].)

The proof of our result relies on the variational principle of Ekeland (see [11]) complemented with the, by now, well-known concentration compactness method of P.L. Lions (see [12]).

One of the advantages in using Ekeland's variational principle is that it allows us to split the geometry of the problem from its compactness aspect. This approach simplifies the one applied by Struwe in [7] for the subcritical case.

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The use of the concentration compactness method to deal with the p-Laplacian has been used by several authors before. One of the first results in this direction was obtained by García Azorero and Peral in [13]. Here, we borrow some ideas from that work.

For related results with subcritical growth, see the already mentioned [7] and more recently [14,15].

Also, for related results with critical growth, but with subcritical power perturbation, see the seminal work of Guedda and Veron [16] and, more recently, the paper by Cingolani and Vannella [17].

Throughout this work, by (weak) solutions of (P) we understand critical points of the associated energy functional acting on the Sobolev space $W_0^{1,p}(\Omega)$:

$$\Phi(v) = \frac{1}{p}\int_\Omega |\nabla v|^p \mathrm{d}x - \frac{1}{p^*}\int_\Omega |u|^{p^*}\mathrm{d}x - \lambda\int_\Omega F(x,v)\mathrm{d}x, \tag{1}$$

where $F(x,u) = \int_0^u f(x,z)\mathrm{d}z$.

We will denote

$$\mathcal{F}_\lambda(v) = \int_\Omega \frac{1}{p^*}|v|^{p^*} + \lambda F(x,v)\mathrm{d}x, \tag{2}$$

so the functional Φ can be rewritten as

$$\Phi(v) = \frac{1}{p}\|v\|^p_{W^{1,p}(\Omega)} - \mathcal{F}_\lambda(v).$$

2. Assumptions and statement of the results

The precise assumptions on the source terms f are as follows:

(F1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$, is a measurable function with respect to the first argument and continuously differentiable with respect to the second argument for almost every $x \in \Omega$. Moreover, $f(x,0) = 0$ for every $x \in \Omega$.

(F2) There exist $q \in (p, p^*)$ and constants $c_1 \in (1/(p^*-1), 1/(p-1))$, $c_2 \in (p, p^*)$, $0 < c_3 < c_4$, such that for any $u \in L^q(\Omega)$,

$$\begin{aligned} c_3\|u\|^q_{L^q(\Omega)} &\le c_2\int_\Omega F(x,u)\mathrm{d}x \le \int_\Omega f(x,u)u\mathrm{d}x \\ &\le c_1\int_\Omega f_u(x,u)u^2\mathrm{d}x \le c_4\|u\|^q_{L^q(\Omega)}. \end{aligned}$$

Remark 1. Observe that this set of hypotheses on the nonlinear term f are weaker than the ones considered by [11].

Remark 2. We exhibit now two families of examples of nonlinearities that fulfill all of our hypotheses.

- $f(x,u) = |u|^{q-2}u + |u_+|^{r-2}u_+$, if $r < q$.
  Hypotheses (F1)–(F2) are clearly satisfied.
- $f(x,u) = \begin{cases} |u|^{q-2}u + |u|^{r_1-2}u & u \ge 0 \\ |u|^{q-2}u + |u|^{r_2-2}u & u < 0 \end{cases}$, if $r_1, r_2 < q$.
  Hypotheses (F1)–(F2) are, again, clearly satisfied.

So the main result of the paper reads:

Theorem 1. *Under assumptions* (F1)–(F2), *there exist $\lambda^* > 0$ depending only on n, p, q and the constant c_3 in* (F2), *such that for every $\lambda > \lambda^*$, there exists three different, nontrivial, (weak) solutions of problem* (P). *Moreover these solutions are, one positive, one negative and the other one has non-constant sign.*

3. Proof of Theorem 1

The proof uses the same approach as in [7]. That is, we will construct three disjoint sets $K_i \neq \emptyset$ not containing 0 such that Φ has a critical point in K_i. These sets will be subsets of C^1-manifolds $M_i \subset W^{1,p}(\Omega)$ that will be constructed by imposing a sign restriction and a normalizing condition.

In fact, let

$$M_1 = \left\{u \in W_0^{1,p}(\Omega) : \int_\Omega u_+ > 0 \text{ and } \int_\Omega |\nabla u_+|^p - |u_+|^{p^*}\mathrm{d}x = \int_\Omega \lambda f(x,u)u_+\mathrm{d}x\right\},$$

$$M_2 = \left\{u \in W_0^{1,p}(\Omega) : \int_\Omega u_- > 0 \text{ and } \int_\Omega |\nabla u_-|^p - |u_-|^{p^*}\mathrm{d}x = -\int_\Omega \lambda f(x,u)u_-\mathrm{d}x\right\},$$

$$M_3 = M_1 \cap M_2$$

where $u_+ = \max\{u, 0\}$, $u_- = \max\{-u, 0\}$ are the positive and negative parts of u, and $\langle\cdot,\cdot\rangle$ is the duality pairing of $W^{1,p}(\Omega)$.