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Approximating the Riemann–Stieltjes integral in terms of generalised trapezoidal rules

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ABSTRACT

Error bounds in approximating the Riemann–Stieltjes integral in terms of some new generalised trapezoidal rules are given. Applications for approximating the Riemann integral of a two-function product are provided as well.

1. Introduction

In [7], the authors considered the following *generalised trapezoid formula*

$$[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a), \quad x\in[a,b]$$

in order to approximate the *Riemann–Stieltjes integral* $\int_a^b f(t)\,du(t)$. They proved the inequality

$$\left|\int_a^b f(t)\,du(t)-[u(b)-u(x)]f(b)-[u(x)-u(a)]f(a)\right|\le H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f) \tag{1.1}$$

for any $x\in[a,b]$, provided that $f:[a,b]\to\mathbb{R}$ is of *bounded variation* on $[a,b]$ and u is of $r-H$*-Hölder type*, i.e., $|u(t)-u(s)|\le H|t-s|^r$ for any $t,s\in[a,b]$, where $r\in(0,1]$ and $H>0$ are given. Here $\bigvee_a^b(f)$ denotes the *total variation* of f on $[a,b]$.

In [4], the following dual result has been obtained as well:

$$\left|\int_a^b f(t)\,du(t)-[u(b)-u(x)]f(b)-[u(x)-u(a)]f(a)\right|\le H\left[(x-a)^r\bigvee_a^x(u)+(b-x)^r\bigvee_x^b(u)\right]$$

$$\le\begin{cases} H\left[(x-a)^r+(b-x)^r\right]\left[\frac{1}{2}\bigvee_a^b(u)+\frac{1}{2}\left|\bigvee_a^x(u)-\bigvee_x^b(u)\right|\right]; \\ \left[(x-a)^{qr}+(b-x)^{qr}\right]^{\frac{1}{q}}\left[\left[\bigvee_a^x(u)\right]^p+\left[\bigvee_x^b(u)\right]^p\right]^{\frac{1}{p}} \quad \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(u) \end{cases} \tag{1.2}$$

for any $x\in[a,b]$, provided that f is of $r-H$-Hölder type and u is of bounded variation.

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For other inequalities of this type, see the recent papers [6,3,5].

The main aim of the present paper is to continue the study on approximating the Riemann–Stieltjes integral $\int_a^b f(t)\,\mathrm{d}u(t)$ by the use of some generalised trapezoid-type rules. To be more specific, we investigate the error bounds in approximating $\int_a^b f(t)\,\mathrm{d}u(t)$ by the simpler quantities:

$$f(b)\left[\frac{1}{b-a}\int_a^b u(t)\,\mathrm{d}t - u(a)\right] + f(a)\left[u(b) - \frac{1}{b-a}\int_a^b u(t)\,\mathrm{d}t\right] \tag{1.3}$$

and

$$[u(b)-u(a)]\left[f(b)+f(a)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right], \tag{1.4}$$

provided the Riemann integral $\int_a^b f(t)\,\mathrm{d}t$ exists and can be either computed exactly or can be accurately approximated by the use of various classical quadrature rules. Applications for approximating the Riemann integral of a two-function product are provided as well.

For other recent quadrature rules for the Riemann–Stieltjes integral, see [2,8,9] and the references therein. For related results, see [1,10–12].

2. Integral representation of the error

For a function $g:[a,b]\to\mathbb{R}$ we define $\Psi_g:[a,b]\to\mathbb{R}$ by

$$\Psi_g(t) := g(t) - \frac{g(a)(t-a)+g(b)(b-t)}{b-a}. \tag{2.1}$$

We can state the following result.

Theorem 1. *If $f,u:[a,b]\to\mathbb{R}$ are bounded on $[a,b]$ and such that the Riemann–Stieltjes integral $\int_a^b f(t)\,\mathrm{d}u(t)$ and the Riemann integral $\int_a^b u(t)\,\mathrm{d}t$ exist, then*

$$\int_a^b f(t)\,\mathrm{d}u(t) - \left\{f(b)\left[\frac{1}{b-a}\int_a^b u(t)\,\mathrm{d}t - u(a)\right] + f(a)\left[u(b)-\frac{1}{b-a}\int_a^b u(t)\,\mathrm{d}t\right]\right\}$$
$$= \int_a^b \Psi_f(t)\,\mathrm{d}u(t). \tag{2.2}$$

Proof. We have

$$\int_a^b \Psi_f(t)\,\mathrm{d}u(t) = \int_a^b f(t)\,\mathrm{d}u(t) - \frac{1}{b-a}\left[f(a)\int_a^b (t-a)\,\mathrm{d}u(t) + f(b)\int_a^b (b-t)\,\mathrm{d}u(t)\right]. \tag{2.3}$$

Integrating by parts in the Riemann–Stieltjes integral, we also have $\int_a^b (t-a)\,\mathrm{d}u(t) = (b-a)\,u(b) - \int_a^b u(t)\,\mathrm{d}t$ and $\int_a^b (b-t)\,\mathrm{d}u(t) = -(b-a)\,u(a) + \int_a^b u(t)\,\mathrm{d}t$. Then, we have

$$\frac{1}{b-a}\left[f(a)\int_a^b (t-a)\,\mathrm{d}u(t) + f(b)\int_a^b (b-t)\,\mathrm{d}u(t)\right]$$
$$= f(b)\left[\frac{1}{b-a}\int_a^b u(t)\,\mathrm{d}t - u(a)\right] + f(a)\left[u(b)-\frac{1}{b-a}\int_a^b u(t)\,\mathrm{d}t\right],$$

and by (2.3) we deduce the desired result (2.2). □

The second result can be stated as:

Theorem 2. *With the assumptions of Theorem 1, we have:*

$$[u(b)-u(a)]\left[f(b)+f(a)-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right] - \int_a^b f(t)\,\mathrm{d}u(t) = \int_a^b \Psi_u(t)\,\mathrm{d}f(t). \tag{2.4}$$