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Computing ODE symmetries as abnormal variational symmetries

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ABSTRACT

We give a new computational method to obtain symmetries of ordinary differential equations. The proposed approach appears as an extension of a recent algorithm to compute variational symmetries of optimal control problems [P.D.F. Gouveia, D.F.M. Torres, Automatic computation of conservation laws in the calculus of variations and optimal control, Comput. Methods Appl. Math. 5 (4) (2005) 387–409], and is based on the resolution of a first order linear PDE that arises as a necessary and sufficient condition of invariance for abnormal optimal control problems. A computer algebra procedure is developed, which permits one to obtain ODE symmetries by the proposed method. Examples are given, and results compared with those obtained by previous available methods.

1. Introduction

Sophus Lie was the first to introduce the use of symmetries into the study of differential equations, Emmy Noether the first to recognize the important role of symmetries in the calculus of variations. Currently, all the computer algebra systems that address differential equations provide several tools to help the user with the analysis of Lie symmetries. Recently, the authors developed a computer algebra package for the automatic computation of Noether variational symmetries in the calculus of variations and optimal control [5], now available as part of the Maple Application Center at http://www.maplesoft.com/applications/app_center_view.aspx?AID=1983.

The omnipresent tools for Lie symmetries provide a great help for the search of solutions of ODEs, their classification, order reduction, proof of integrability, or in the construction of first integrals. From the mathematical point of view, a ODE symmetry is described by a group of transformations that keeps the ordinary differential equation invariant. Depending on the type of transformations one is considering, different symmetries are obtained. An important class of symmetries is obtained considering a one-parameter family of transformations, which form a local Lie group. Those transformations are often represented by a set of functions known as the infinitesimal generators. From the practical point of view, the determination of the infinitesimal generators that define a symmetry for a given ODE is, in general, a complex task [6,11]. To address the problem, we follow a different approach.

We propose a new method for computing symmetries of ODEs by using a Noetherian perspective. Making use of our previous algorithm [5], that has shown up good results for the computation of Noether variational symmetries of problems of the calculus of variations and optimal control, we look to an ODE as being the control system of an optimal control problem. Then, we obtain symmetries for the ODE by computing the abnormal variational symmetries of the associated optimal control problem.

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This paper is organized as follows. In Section 2, the necessary concepts associated with variational symmetries in optimal control are reviewed. The new method for computing symmetries of ODEs is explained in Section 3. The method is illustrated in Section 4, where we compute symmetries for three distinct ODEs and compare the results with the ones obtained by the standard procedures available in `Maple`. We end the paper with Section 5 of conclusions and final comments. The definitions of the new `Maple` procedure that implements our method are given in Appendix.

2. Symmetries in optimal control

Without loss of generality, we consider the optimal control problem in Lagrange form: to minimize an integral functional

$$I[\mathbf{x}(\cdot), \mathbf{u}(\cdot)] = \int_a^b L(t, \mathbf{x}(t), \mathbf{u}(t))\, \mathrm{d}t \tag{1}$$

subject to a control system described by a system of ordinary differential equations of the form

$$\dot{\mathbf{x}}(t) = \boldsymbol{\varphi}(t, \mathbf{x}(t), \mathbf{u}(t)), \tag{2}$$

together with appropriate boundary conditions. The Lagrangian $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and the velocity vector $\boldsymbol{\varphi} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are assumed to be continuously differentiable functions with respect to all their arguments. The controls $\mathbf{u} : [a, b] \to \mathbb{R}^m$ are piecewise continuous functions; the state variables $\mathbf{x} : [a, b] \to \mathbb{R}^n$ continuously differentiable functions.

The celebrated Pontryagin Maximum Principle [10] (PMP for short) gives a first-order necessary optimality condition. The PMP can be proved from a general Lagrange multiplier theorem. One introduces the Hamiltonian function

$$H(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = \psi_0 L(t, \mathbf{x}, \mathbf{u}) + \boldsymbol{\psi}^{\mathrm{T}} \cdot \boldsymbol{\varphi}(t, \mathbf{x}, \mathbf{u}), \tag{3}$$

where $(\psi_0, \boldsymbol{\psi}(\cdot))$ are the "Lagrange multipliers", with $\psi_0 \le 0$ a constant and $\boldsymbol{\psi}(\cdot)$ a n-vectorial piecewise C^1-smooth function, and the multiplier theorem asserts that the optimal control problem is equivalent to the maximization of the augmented functional

$$J[\mathbf{x}(\cdot), \mathbf{u}(\cdot), \psi_0, \boldsymbol{\psi}(\cdot)] = \int_a^b \left(H(t, \mathbf{x}(t), \mathbf{u}(t), \psi_0, \boldsymbol{\psi}(t)) - \boldsymbol{\psi}(t)^{\mathrm{T}} \cdot \dot{\mathbf{x}}(t)\right) \mathrm{d}t. \tag{4}$$

Definition 1. A quadruple $(\mathbf{x}(\cdot), \mathbf{u}(\cdot), \psi_0, \boldsymbol{\psi}(\cdot))$ satisfying the Pontryagin Maximum Principle is said to be a (Pontryagin) *extremal*. An extremal is said to be *normal* when $\psi_0 \neq 0$, *abnormal* when $\psi_0 = 0$.

Let $\mathbf{h}^s : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ be a one-parameter group of $\mathbb{C}^1$ transformations of the form

$$\mathbf{h}^s(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = (h_t^s(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}), \mathbf{h}_{\mathbf{x}}^s(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}), \mathbf{h}_{\mathbf{u}}^s(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}), \mathbf{h}_{\boldsymbol{\psi}}^s(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi})). \tag{5}$$

Without loss of generality, we assume that the identity transformation of the group (5) is obtained when the parameter s is zero:

$$h_t^0(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = t, \qquad \mathbf{h}_{\mathbf{x}}^0(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = \mathbf{x},$$
$$\mathbf{h}_{\mathbf{u}}^0(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = \mathbf{u}, \qquad \mathbf{h}_{\boldsymbol{\psi}}^0(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = \boldsymbol{\psi}.$$

Associated with a one-parameter group of transformations (5), we introduce its *infinitesimal generators*:

$$T(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = \left.\frac{\partial}{\partial s} h_t^s\right|_{s=0}, \qquad \mathbf{X}(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = \left.\frac{\partial}{\partial s} \mathbf{h}_{\mathbf{x}}^s\right|_{s=0},$$
$$\mathbf{U}(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = \left.\frac{\partial}{\partial s} \mathbf{h}_{\mathbf{u}}^s\right|_{s=0}, \qquad \boldsymbol{\Psi}(t, \mathbf{x}, \mathbf{u}, \psi_0, \boldsymbol{\psi}) = \left.\frac{\partial}{\partial s} \mathbf{h}_{\boldsymbol{\psi}}^s\right|_{s=0}. \tag{6}$$

We can define variational invariance using the augmented functional (4) and the one-parameter group of transformations (5) or an equivalent condition in terms of the generators (6):

Definition 2 (*[3,13]*). An optimal control problem (1)–(2) is said to be *invariant* under (6) or, equivalently, (6) is said to be a *symmetry* of the problem (1)–(2) if

$$\frac{\partial H}{\partial t} T + \frac{\partial H}{\partial \mathbf{x}} \cdot \mathbf{X} + \frac{\partial H}{\partial \mathbf{u}} \cdot \mathbf{U} + \frac{\partial H}{\partial \boldsymbol{\psi}} \cdot \boldsymbol{\Psi} - \boldsymbol{\Psi}^{\mathrm{T}} \cdot \dot{\mathbf{x}} - \boldsymbol{\psi}^{\mathrm{T}} \cdot \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} + H \frac{\mathrm{d}T}{\mathrm{d}t} = 0, \tag{7}$$

with H the Hamiltonian (3).

A computational algorithm to obtain the infinitesimal generators T, $\mathbf{X}$, $\mathbf{U}$, and $\boldsymbol{\Psi}$ that form a variational symmetry (7) for a given optimal control problem (1)–(2) was developed in [5]. Here we remark that the abnormal variational symmetries (*i.e.* the ones associated with $\psi_0 = 0$) obtained by the method introduced in [5] provide symmetries for ordinary differential equations.