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Geometrical unified theory of Rikitake system and KCC-theory

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ABSTRACT

The Rikitake system as nonlinear dynamical systems in geomagnetism can be studied based on the KCC-theory and the unified field theory. Especially, the behavior of the magnetic field of the Rikitake system is represented in the electrical system projected from the electro-mechanical unified system. Then, the KCC-invariants for the electrical and mechanical systems can be obtained. The third invariant as the torsion tensor expresses the aperiodic behavior of the magnetic field. Moreover, as a result of the projection, a protrusion between the mechanical and electrical systems is represented by the Euler–Schouten tensor. This Euler–Schouten tensor and the third invariant consist of the same mutual-inductance. Therefore, the aperiodic behavior of the magnetic field can be characterized by the protrusion between the electrical and mechanical systems.

1. Introduction

Nonlinear dynamical systems appear in various geophysical phenomena. For example, in geomagnetism, the behavior of the Earth's magnetic field has been reversed aperiodically. The mechanism of generating the geomagnetic field is explained by the dynamo theory [26]. Especially, a simple dynamo model called the Rikitake system [30] has been proposed to describe the behavior of an aperiodic magnetic field. The Rikitake system consists of conductive disks which rotate in the magnetic field, i.e. the Rikitake system can be mechanically regarded as a rotating gyrostat [13]. Moreover, the dynamics of a rotating rigid body relates to nonlinear geophysical fluid dynamics [13,14,27]. In particular, a nonlinear dynamical system called the Lorenz system [24] is derived from the gyrostat system [13,14]. Mathematically, for these simplified models, the equations of motion are uniquely expressed by a system of nonlinear ordinary differential equations [31]. Therefore, the Lorenz system and the Rikitake system can be discussed in the same theoretical method.

The nonlinear equations of motion cannot be solved analytically and so geometrical approaches have been studied [5]. Generally, equations of motion of the nonlinear dynamical systems are regarded as a system of second order differential equations. In this case, the nonlinear dynamical systems can be geometrically investigated by the general path-space theory of Kosambi [22], Cartan [10] and Chern [11] (KCC-theory). There are five KCC-invariants [3] and the stability of the nonlinear dynamical systems has been discussed in ecology [1,2,4]. For the Rikitake system, it has been studied [32–34] and the chaotic behavior expressed by the KCC-invariants in the framework of the Finsler and the Lagrange geometries [4,8,25]. Moreover, the Lorenz system can be considered in the Riemann–Cartan space [28,29]. Therefore, aperiodic trajectories of the dynamical systems can be expressed by the geometric invariants of the KCC-theory.

From another viewpoint, the nonlinear dynamical systems can be regarded as unified systems which consist of different physical fields. In this case, the chaotic behavior of dynamical systems is described in a subspace projected from the unified

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system. For example, the Rikitake system is a unified system of the mechanical and electrical systems [23] and so the aperiodic reversals of the magnetic field are expressed in the electrical system projected from the unified system. This concept of the unified dynamical system is similar to the theory of the unified field (e.g. [19,20]). Geometrically, such a unified field theory has been discussed [6,7,16,17] from the vector bundle-like standpoint and the method of contact tensor calculus [35,36]. The unified field is decomposed into the non-holonomic subspaces, i.e. one of the subspaces is composed of the geometrical degrees of freedom and the other consists of the physical ones. Then, the mapping relations between these subspaces constitute the interaction field. Therefore, the aperiodic behavior of the nonlinear dynamical systems can be discussed based on the interaction between different physical fields. In the following part of this paper, we concretely study the aperiodic behavior of magnetic field in the Rikitake system.

In Section 2, we briefly review the geometric theory (the KCC-theory). Then, in Section 3, we give a unified viewpoint of the nonlinear dynamical systems. In Sections 4 and 5, the KCC-theory is applied to the Rikitake system and the aperiodic behavior of the magnetic field is discussed geometrically.

2. Geometric preliminary (KCC-theory)

In this section, the geometric background of this study is briefly explained based on the papers [1–4,8]. Throughout this paper, Einstein's summation convention is used.

Let $M(x)$ be a real smooth n-dimensional manifold called a base space. A point $x \in M(x)$ has local coordinates (x^i), where $i = 1, \dots, n$. We also call the base space as the "(x)-field". Then, let $M(y)$ be a real smooth n-dimensional manifold called a phase space spanned by the local coordinates $(y^i) = (\mathrm{d}x^i/\mathrm{d}t)$, where the parameter t is a time which is regarded as an absolute invariant. We call the phase space as the "(y)-field". The tangent bundle over $M(x)$ is denoted by $TM(x)$. The local chart of a point $Z \in TM(x)$ is denoted by $(Z^I) = (x^i, y^i)$, where $I = 1, \dots, 2n$. Then, a non-singular coordinate transformation is considered as

$$\tilde{t} = t, \qquad \tilde{x}^i = \tilde{x}^i(x^1, x^2, \dots, x^n). \tag{1}$$

Then, a local expression of the vector field called a semispray S on $TM(x)$ is

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x^j, y^j)\frac{\partial}{\partial y^i}, \tag{2}$$

where the smooth function $G^i(x^j, y^j)$ is defined on domains of the local chart. An n-dimensional distribution called nonlinear connection $N : Z \in TM(x) \mapsto N_Z \subset T_Z TM(x)$ is supplementary to the vertical distribution V, i.e. $T_Z TM(x) = N_Z \oplus V_Z$. An adapted basis and its dual basis (Berwald basis [3]) to this direct sum are denoted by $(\delta/\delta x^i, \partial/\partial y^i)$, where

$$\left(\frac{\partial}{\partial Z^I}\right) = \left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right) = \left(\frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right), \tag{3}$$

$$(\mathrm{d}Z^I) = (\mathrm{d}x^i, \delta y^i) = (\mathrm{d}x^i, \mathrm{d}y^i + N_j^i \mathrm{d}x^j). \tag{4}$$

The coefficients of nonlinear connection induced by the semispray are defined by $N_j^i \equiv \partial G^i/\partial y^j$.

A behavior of nonlinear dynamical systems is expressed by a path $c(t) = (x^i(t))$ of the semispray S described by

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + 2G^i(x^j, y^j) = 0. \tag{5}$$

Eq. (5) can also be rewritten as the Euler–Lagrange equation [8,25]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial y^i}\right) - \frac{\partial \mathcal{L}}{\partial x^i} = E_i, \tag{6}$$

where $\mathcal{L}$ is the Lagrangian and E_i is the external force.

With respect to the Berwald basis, a Finsler connection called the Berwald connection $B\Gamma = (G_{jk}^i, N_j^i, C_{jk}^i)$ has local coefficients [4]:

$$G_{jk}^i(x^l) = \frac{\partial N_j^i}{\partial y^k} = \frac{\partial^2 G^i}{\partial y^j \partial y^k}, \qquad C_{jk}^i = 0. \tag{7}$$

In the following, for the application to physics, we consider Eq. (5) with the Berwald connection and the additional terms:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + G_{jk}^i(x^l)\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t} + \gamma_j^i(x^k)\frac{\mathrm{d}x^j}{\mathrm{d}t} + f^i = 0, \tag{8}$$

where $\gamma_j^i \mathrm{d}x^j/\mathrm{d}t$ and f^i express certain additional forces.