Nonlinear Analysis 71 (2009) e243–e258

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Conditional Lie Bäcklund symmetries of Hamilton–Jacobi equations

Changzheng Qu[a,b,*], Lina Ji[a,c]

[a] *Center for Nonlinear Studies, Northwest University, Xi'an, 710069, PR China*

[b] *Department of Mathematics, Northwest University, Xi'an, 710069, PR China*

[c] *Department of Information & Computational Science, He'nan Agricultural University, Zhengzhou, 450002, PR China*

A B S T R A C T

The conditional Lie Bäcklund symmetry method, as a generalization of the conditional symmetry and Lie Bäcklund symmetry methods, is developed to study Hamilton–Jacobi equations. It is shown that two Hamilton–Jacobi equations admit conditional Lie Bäcklund symmetries of second-order for certain smooth functions. As a result, a complete description of the structure of solutions to the resulting equations associated with the conditional Lie Bäcklund symmetries is performed.

1. Introduction

In this paper, we study conditional Lie Bäcklund symmetries (CLBSs) of the Hamilton–Jacobi equation

$$u_t = f(x, u, u_x), \tag{1.1}$$

where $f(x, u, u_x)$ is a sufficiently smooth given function of x, u and u_x. It is well-known that Hamilton–Jacobi equations have a wide range of applications such as classical mechanics, quantum mechanics and contemporary problems of control etc. There are many different applications of the well-developed mathematical theory on Hamilton–Jacobi equations concerning construction of solutions, uniqueness and existence of viscosity solutions, singularities and discontinuity and other aspects of the qualitative properties of the solutions [1–4]. It is worth noting that Hamilton–Jacobi equations can be used to describe the long time behavior, blow up profile and geometric properties of nonlinear diffusion equations [5–8]. Thus, a knowledge of the properties of solutions to Hamilton–Jacobi equations will be helpful to explore properties of nonlinear diffusion equations.

Two special cases of (1.1) are

$$u_t = u_x^{n+1} + B(u)u_x + C(u), \quad t \in \mathbb{R}^+, x \in \mathbb{R}, n \neq -1, \tag{1.2}$$

and

$$u_t = u_x^{m+2} + p(x)B(u)u_x^{m+1} + Q(x, u), \quad t \in \mathbb{R}^+, x \in \mathbb{R}, m \neq -1, -2, \tag{1.3}$$

where $B(u)$, $C(u)$ and $Q(x, u)$ are sufficiently smooth functions of the indicated variables.

The ordinary method of constructing solutions of first-order PDEs is the method of characteristics [4]. The general solutions to linear and quasilinear first-order PDEs can be obtained by integrating the corresponding characteristic ODEs. But for the fully nonlinear first-order PDEs, the method of characteristics can be used only to solve the initial or boundary value problem. Thus, it is interesting to develop other methods to solve the fully nonlinear first-order PDEs.

* Corresponding address: Center for Nonlinear Studies, Northwest University, No. 229, Taibai North Road, Shaan Xi, Xi'an, 710069, PR China. Tel.: +86 29 88303548; fax: +86 29 88303401.

E-mail address: czqu@nwu.edu.cn (C.Z. Qu).

0362-546X/$ – see front matter
doi:10.1016/j.na.2008.10.045

It is well-known that symmetry group related methods have been proved to be effective to seek symmetry reductions and exact solutions of nonlinear partial differential equations (PDEs). Many interesting results for qualitative analysis of nonlinear PDEs are associated with the properties of exact solutions obtained through the symmetry group methods. For example, the Pohazaev identity for nonlinear elliptic equations arises from the scaling invariance of the equations [9]. The classical method for finding symmetry reductions of PDEs is the Lie group method of infinitesimal transformations [10–12]. Several generalizations to the classical method have been introduced. These include the nonclassical method [13], the direct method [14–17] and the CLBS method [18–31]. Those methods have been successfully applied to obtain solutions and conservation laws of nonlinear PDEs, and the solutions obtained by the symmetry group methods can be used to explore various properties of nonlinear PDEs [5,6].

The purpose of this paper is to explore symmetries of the Hamilton–Jacobi equations (1.2)–(1.3). The approach used here is the CLBS method. The layout of the paper is organized as follows. In Section 2, we present some necessary notions, definitions and fundamental theorems on CLBSs. In Section 3, we discuss Hamilton–Jacobi equation (1.2) admitting CLBSs with the characteristic

$$\eta = u_{xx} + H(u)u_x^2 + F(u)u_x + G(u). \tag{1.4}$$

Eq. (1.3) admitting CLBSs with the characteristic

$$\eta = u_{xx} + H(u)u_x^2 + F(x,u)u_x^{2-n} + G(x,u)u_x^{1-n} \tag{1.5}$$

are discussed in Section 4. In Section 5, we study the equation (1.3) admitting CLBs (1.4). Section 6 contains a concluding remarks on this work.

2. Conditional Lie Bäcklund symmetry method

Let us review the basic facts on CLBS of nonlinear evolution equations.

Suppose that the n-th order nonlinear evolution equation of the form

$$u_t = E(x, t, u, u_1, \ldots, u_m), \quad t \in \mathbb{R}^+, x \in \mathbb{R}, \tag{2.1}$$

where $u_j = \partial^j u/\partial x^j$, is invariant under a non-Lie point group of infinitesimal transformations

$$\begin{aligned}
&u' = u + \epsilon\eta(t, x, u, u_1, \ldots, u_N) + O(\epsilon^2),\\
&u'_t = u_t + \epsilon D_t\eta(t, x, u, u_1, \ldots, u_N) + O(\epsilon^2),\\
&u'_x = u_x + \epsilon D_x\eta(t, x, u, u_1, \ldots, u_N) + O(\epsilon^2),\\
&\cdots.
\end{aligned}$$

The above group is generated by an evolutionary vector field with η as its characteristic

$$V = \sum_{k=0}^{\infty} D_x^k\eta \frac{\partial}{\partial u_k}, \tag{2.2}$$

where we use the following notations

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{k+1}\frac{\partial}{\partial u_k}, \qquad D_x^{j+1} = D_x(D_x^j), \quad D_x^0 = 1.$$

Definition 2.1. The evolutionary vector field (2.2) is said to be a Lie-Bäcklund symmetry of (2.1) if and only if

$$V^{(m)}(u_t - E)|_L = 0,$$

where and hereafter $V^{(m)}$ is the m-th prolongation of V and L is the set of all differential consequences of the equation, that is

$$u_t - E = 0, \qquad D_x^j D_t^k(u_t - E) = 0, \quad j, k = 0, 1, 2, \ldots.$$

Definition 2.2. The evolutionary vector field (2.2) is said to be a CLBS of (2.1) if and only if

$$V^{(m)}(u_t - E)|_{L\cap M} = 0,$$

where M denotes the set of all differential consequences of equation $\eta = 0$ with respect to x, that is $D_x^j\eta = 0, j = 0, 1, 2, \ldots.$