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On a comparison principle for delay coupled systems with nonlocal and nonlinear boundary conditions[☆]

Jin Liang [a], Huan-Yu Wang [b], Ti-Jun Xiao [c,*]

[a] Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, PR China
[b] Department of Mathematics, Nanjing Agricultural University, Nanjing 210095, PR China
[c] School of Mathematical Sciences, Fudan University, Shanghai 200433, PR China

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ABSTRACT

This paper is concerned with delay coupled systems of parabolic equations with nonlocal and nonlinear boundary conditions. For them, a new and general comparison principle is established, which is more general and useful than the existing results.

1. Introduction

It is well known that various comparison principles play key roles in the study of many mathematical problems; cf., e.g. [1–7,9,10]. Stimulated by these works, in this paper, we focus our attention on establishing a new comparison principle for a class of delay coupled systems of parabolic equations with *nonlocal and nonlinear* boundary conditions. As one will see, our results improve many previous results since the boundary conditions involved here are nonlinear and the constant M (in (H2) below) could be bigger than 1 when the volume of Ω is bigger than 1.

Throughout this paper, we let Ω be a bounded domain with the boundary $\partial\Omega$ in R^n $(n \geq 1)$ and $\frac{\partial}{\partial\nu}$ denote the outward normal derivative on $\partial\Omega$. Let $T, \tau_i > 0$ $(i = 1, \ldots, N)$ and set:

$$\begin{aligned}
&D_T := \Omega \times (0, T),\\
&\bar{D}_T := \bar{\Omega} \times [0, T],\\
&\bar{Q}_T^{(i)} := \bar{\Omega} \times [-\tau_i, T] \quad (i = 1, \ldots, N),\\
&\bar{Q}_T := \bar{Q}_T^{(1)} \times \cdots \times \bar{Q}_T^{(N)},\\
&S_T := \partial\Omega \times (0, T),\\
&\mathbf{u} = \mathbf{u}(x, t) := (u_1(x, t), \ldots, u_N(x, t)),\\
&\mathbf{u}_\tau = \mathbf{u}_\tau(x, t) = (u_1(x, t - \tau_1), \ldots, u_N(x, t - \tau_N)) := (u_{\tau_1}(x, t), \ldots, u_{\tau_N}(x, t)).
\end{aligned}$$

Moreover, $\mathbf{C}^{1,2}(D_T)$ denotes the set of functions which are once continuously differentiable in t and twice continuously differentiable in x for $(x, t) \in D_T$.

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* Corresponding author.
E-mail addresses: jinliang@sjtu.edu.cn (J. Liang), huanyuw@njau.edu.cn (H.-Y. Wang), xiaotj@ustc.edu.cn, tjxiao@fudan.edu.cn (T.-J. Xiao).

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Of concern is the following delay coupled system of parabolic equations with nonlocal and nonlinear boundary conditions

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - L_i u_i = f_i(x,t,\mathbf{u},\mathbf{u}_\tau) & (t>0, x\in\Omega, i=1,\dots,N), \\ B_i u_i = \displaystyle\int_\Omega K_i(x,y,t,\mathbf{u}(y,t))\mathrm{d}y & (t>0,\ x\in\partial\Omega,\ i=1,\dots,N), \\ u_i(x,t)=\eta_i(x,t) & (-\tau_i\le t\le 0,\ x\in\Omega,\ i=1,\dots,N), \end{cases} \tag{1.1}$$

where the operators L_i and B_i are defined by

$$L_i = \sum_{j,k=1}^n a_{jk}^{(i)}(x,t)\frac{\partial^2}{\partial x_j\partial x_k} + \sum_{j=1}^n b_j^{(i)}(x,t)\frac{\partial}{\partial x_j} \quad (i=1,\dots,N),$$

$$B_i = \alpha_i(x,t)\frac{\partial}{\partial \nu} + 1 \quad (\alpha_i\ge 0,\ i=1,\dots,N),$$

the functions $f_i(\cdot,\mathbf{u},\mathbf{v})$ are continuously differentiable in $\mathbf{u}$ and $\mathbf{v}$, and the functions $K_i(x,y,t,\mathbf{u})$ are continuously differentiable in $\mathbf{u}$ and suitably smooth in $x,y,t,\ i=1,\dots,N$.

Write

$$K_{i(j)} := \frac{\partial K_i}{\partial u_j}, \quad i,j=1,2,\dots,N.$$

The following hypotheses will be used in the next section.

(H1)

$$K_{i(j)}(x,y,t,\mathbf{u}(y,t))\ge 0,$$

$$\int_\Omega \sum_{j=1}^N K_{i(j)}(x,y,t,\mathbf{u}(y,t))\mathrm{d}y < 1, \quad x\in\partial\Omega,\ y\in\Omega,\ t>0;$$

(H2) There is $M>0$ such that $0\le K_{i(j)}(x,y,t,\mathbf{u}(y,t))\le M,\ x\in\partial\Omega,\ y\in\Omega,\ t>0$.

2. Results and proofs

For the reader's convenience, we recall two basic concepts first.

Definition 2.1. Write $\mathbf{u},\ \mathbf{v}$ in the split form,

$$\mathbf{u} = (u_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i}), \qquad \mathbf{v} = ([\mathbf{v}]_{c_i}, [\mathbf{v}]_{d_i}).$$

The vector function $\mathbf{f}(\cdot,\mathbf{u},\mathbf{v})$ is said to be mixed quasi-monotone in some subset $\mathcal{S}$ of $R^N\times R^N$, if for each $i=1,\dots,N$, there exist nonnegative integers a_i, b_i, c_i and d_i with

$$a_i+b_i=N-1, \qquad c_i+d_i=N$$

such that $f_i(\cdot,u_i,[\mathbf{u}]_{a_i},[\mathbf{u}]_{b_i},[\mathbf{v}]_{c_i},[\mathbf{v}]_{d_i})$ is nondecreasing with respect to the components of $[\mathbf{u}]_{a_i}$ and $[\mathbf{v}]_{c_i}$, and is nonincreasing with respect to the components of $[\mathbf{u}]_{b_i}$ and $[\mathbf{v}]_{d_i}$ for all $(\mathbf{u},\mathbf{v})\in\mathcal{S}$. In particular, $\mathbf{f}(\cdot,\mathbf{u},\mathbf{v})$ is said to be quasi-monotone nondecreasing in $\mathcal{S}$ if $b_i=d_i=0$ for $i=1,\dots,N$.

Lemma 2.2. *Let* (H_1) *hold, and* $b_{ij}+c_{ij}$ $(i,j=1,\dots,N)$ *be bounded in* D_T *such that*

$$b_{ij}+c_{ij}\ge 0 \quad (i\ne j,\ i,j=1,\dots,N).$$

Let $\varphi_i(x,t)\ge\psi_i(x,t)$ $(i=1,\dots,N)$ *and* $\mathbf{u}=(u_1,\dots,u_N),\ \mathbf{v}=(v_1,\dots,v_N)\in(\mathbf{C}^{1,2}(D_T)\cap\mathbf{C}(\bar{D}_T))^N$ *satisfy*

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - L_i u_i \ge \displaystyle\sum_{j=1}^N b_{ij}u_j - \sum_{j=1}^N c_{ij}v_j + \varphi_i(x,t), & (x,t)\in D_T, \\ \dfrac{\partial v_i}{\partial t} - L_i v_i \le \displaystyle\sum_{j=1}^N b_{ij}v_j - \sum_{j=1}^N c_{ij}u_j + \psi_i(x,t), & (x,t)\in D_T, \\ B_i u_i \ge \displaystyle\int_\Omega K_i(x,y,t,\mathbf{u}(y,t))\mathrm{d}y, & (x,t)\in S_T, \\ B_i v_i \le \displaystyle\int_\Omega K_i(x,y,t,\mathbf{v}(y,t))\mathrm{d}y, & (x,t)\in S_T, \\ u_i(x,0)\ge v_i(x,0), & x\in\Omega, \end{cases} \tag{2.1}$$

for every $i=1,\dots,N$*. Then* $\mathbf{u}(x,t)\ge\mathbf{v}(x,t)$ *in* $\overline{D}_T$.