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A multiplicity result for generalized Laplacian systems with multiparameters

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ABSTRACT

In this paper, we consider the existence, nonexistence and multiplicity of a positive solution for a Gelfand type generalized Laplacian system with a singular indefinite weight and a vector parameter. By using the upper and lower solution method and fixed point index theory, we obtain a global multiplicity result with respect to the parameter.

1. Introduction

In this paper, we consider the following nonlinear eigenvalue problem

$$\begin{cases} \Phi(\boldsymbol{u}')' + \boldsymbol{\lambda h}(t)\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{0}, \quad t \in (0,1), \\ \boldsymbol{u}(0) = \boldsymbol{0} = \boldsymbol{u}(1), \end{cases} \tag{P}$$

where $\boldsymbol{u} = (u_1, \dots, u_n)$, $\Phi(\boldsymbol{u}) = (\varphi(u_1), \dots, \varphi(u_n))$, $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism, $\boldsymbol{\lambda} = \operatorname{diag}[\lambda_1, \dots, \lambda_n] \simeq (\lambda_1, \dots, \lambda_n)$, $\boldsymbol{h}(t) = \operatorname{diag}[h_1(t), \dots, h_n(t)] \simeq (h_1(t), \dots, h_n(t))$ and $\boldsymbol{f}(\boldsymbol{u}) = (f_1(u_1, \dots, u_n), \dots, f_n(u_1, \dots, u_n))$. Problem (P) can be rewritten as

$$\begin{cases} \varphi(u_1')' + \lambda_1 h_1(t) f_1(u_1, \dots, u_n) = 0, \quad t \in (0,1), \\ \vdots \\ \varphi(u_n')' + \lambda_n h_n(t) f_n(u_1, \dots, u_n) = 0, \\ u_1(0) = \cdots = u_n(0) = 0 = u_1(1) = \cdots = u_n(1). \end{cases}$$

We denote $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_+^n = \prod_1^n \mathbb{R}_+$ and assume that $h_i \in C((0,1), (0,\infty))$ may be singular at the boundary, $f_i \in C(\mathbb{R}_+^n, (0,\infty))$ satisfying $f_i(0, \dots, 0) > 0$, $i = 1, \dots, n$ and φ satisfying the following condition.

(A_1) For $\sigma > 0$, there exists a constant $C_\sigma > 0$ such that

$$\frac{\varphi(\sigma s)}{\varphi(s)} < C_\sigma \quad \text{for all } s \in \mathbb{R}.$$

Note that φ covers cases $\varphi(x) = |x|^{p-2}x$, $p > 1$ and $\varphi(x) = x$. Moreover, we may take for $x \geq 0$, $\varphi(x) = x^2 + x$ or $\varphi(x) = \ln(x+1)$, as examples. For p-Laplacian or for even more general function φ, problem (P) has been recently

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studied in [4,11]. They proved several results for existence and multiplicity under the various assumptions with respect to $f_0^i = \lim_{\|\boldsymbol{u}\|\to 0} \frac{f_i(\boldsymbol{u})}{\varphi(\|\boldsymbol{u}\|)}$ and $f_\infty^i = \lim_{\|\boldsymbol{u}\|\to \infty} \frac{f_i(\boldsymbol{u})}{\varphi(\|\boldsymbol{u}\|)}$, where $\|\boldsymbol{u}\| = \sum_{i=1}^n |u_i|$. In particular, when the parameter is scalar i.e. $\lambda_1 = \cdots = \lambda_n = \lambda$ and indefinite weights h_i are continuous, the author of [11] proved under assumptions $\sum_{i=1}^n f_0^i = \infty = \sum_{i=1}^n f_\infty^i$ that there exists $\lambda_0 > 0$ such that problem (P) has at least two positive solutions for $\lambda \in (0, \lambda_0)$. For this result, he assumed that φ satisfies that there exist two increasing homeomorphisms ψ_1 and ψ_2 of $(0,\infty)$ onto $(0,\infty)$ such that

$$\psi_1(\sigma)\varphi(s) \le \varphi(\sigma s) \le \psi_2(\sigma)\varphi(s), \tag{1}$$

for $\sigma,\ s > 0$. We notice that condition (A_1) is more general than condition (1).

In this paper, we focus on the case that h_i may be singular at the boundary and the parameter is a vector and then we obtain a global result with respect to the given parameter. More precisely, assume (A_1). Also assume

(A_2) $\int_0^{\frac12} \varphi^{-1}\left(\int_s^{\frac12} h_i(\tau)\mathrm{d}\tau\right)\mathrm{d}s + \int_{\frac12}^1 \varphi^{-1}\left(\int_{\frac12}^s h_i(\tau)\mathrm{d}\tau\right)\mathrm{d}s < \infty, \quad i = 1,\ldots,n,$

(A_3) $f_\infty^i = \infty, i = 1,\ldots,n,$

(A_4) $f_i(u_1,\ldots,u_n) \le f_i(v_1,\ldots,v_n)$, whenever $u_i = v_i$ and $u_j \le v_j, i \ne j,$

then there exists an $(n-1)$-dimensional continuous manifold Γ splitting $\mathbb{R}_+^n \setminus \{\mathbf{0}\}$ into two disjoint subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that problem (P) has at least two positive solutions for $\boldsymbol{\lambda} \in \mathcal{O}_1$, at least one positive solution for $\boldsymbol{\lambda} \in \Gamma$, and no solution for $\boldsymbol{\lambda} \in \mathcal{O}_2$.

As mentioned, we consider the case $\boldsymbol{f}(\mathbf{0}) > \mathbf{0}$ in this work, where the inequality on $\mathbb{R}_+^n$ can be understood componentwise. Even though we expect a similar result for the case $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$ in addition with condition $f_0^i = \infty, i = 1,\ldots,n$, the analysis does not follow a similar way especially in the multiplicity part of the result. We leave a question for this case.

As an application, we obtain existence, nonexistence and multiplicity of positive radial solutions for the following p-Laplacian system on an exterior domain

$$\begin{cases} \Delta_p \boldsymbol{u} + \boldsymbol{\lambda}\boldsymbol{K}(|x|)\boldsymbol{f}(\boldsymbol{u}) = \mathbf{0} & \text{in } \Omega, \\ \boldsymbol{u}(x) = \mathbf{0} & \text{if } |x| = r_0, \\ \boldsymbol{u}(x) \to \mathbf{0} & \text{as } |x| \to \infty, \end{cases} \tag{P_E}$$

where $\Delta_p \boldsymbol{u} = \left(\mathrm{div}(|\nabla u_1|^{p-2}\nabla u_1),\ldots,\mathrm{div}(|\nabla u_n|^{p-2}\nabla u_n)\right)$, $\Omega = \{x \in \mathbb{R}^N : |x| > r_0\}$, $r_0 > 0$, $N > p > 1$, $\boldsymbol{K}(r) = (K_1(r),\ldots,K_n(r))$ and $K_i : [r_0,\infty) \to (0,\infty)$, $i = 1,\ldots,n$ are continuous. For recent works of N-dimensional p-Laplacian systems with indefinite weights, one may refer to De Napoli and Pinasco [2] on a bounded domain, O'Regan and Wang [10] and when $p = 2$, Lee [5], do O et al. [8] on an annular domain, do O et al. [9] on an exterior domain, Chaib [1] on a whole domain and references therein. In particular, do O et al. [9] proved under assumptions $f_0^i = 0, i = 1,\ldots,n$ and $\sum_{i=1}^n f_\infty^i = \infty$ that problem (P_E) has at least one positive radial solution for all $\boldsymbol{\lambda} > \mathbf{0}$. They considered, from the assumption, $\boldsymbol{f}(\mathbf{0}) = \mathbf{0}$ case so that our work is complimentary and the result is new.

For proofs, we mainly make use of the upper and lower solution method for existence and the fixed point index theory for multiplicity.

For vectors, we will use bold characters with no further mention, for example, $\boldsymbol{\lambda}^*$ and $\boldsymbol{a} - \varepsilon$ denote (column) vectors $(\lambda_1^*,\ldots,\lambda_n^*)$ and $(a_1 - \varepsilon,\ldots,a_n - \varepsilon)$ respectively and $\boldsymbol{F}(t,\boldsymbol{u})$ denotes a (column) vector valued function $(F_1(t,u_1,\ldots,u_n),\ldots,F_n(t,u_1,\ldots,u_n))$. We also denote, for example, the ball of radius ε centered at $\boldsymbol{a}$ by $B_\varepsilon(\boldsymbol{a})$.

This paper is organized as follows: In Section 2, we prove a theorem of upper and lower solution method for singular generalized Laplacian systems and introduce a well-known fixed point index theorem for later use. In Section 3, we prove the existence and nonexistence parts of the main result. In Section 4, we prove the global multiplicity of the positive solution for problem (P) by using fixed point index arguments. Finally, in Section 5, we apply the main result to a system of p-Laplacian defined on an exterior domain.

2. Preliminaries

In this section, we give and prove a fundamental theorem of the upper and lower solution method for singular generalized Laplacian systems and introduce a fixed point index theorem for later use. Consider

$$\begin{cases} \Phi(\boldsymbol{u}')' + \boldsymbol{F}(t,\boldsymbol{u}) = \mathbf{0}, & t \in (0,1), \\ \boldsymbol{u}(0) = \boldsymbol{a}, \quad \boldsymbol{u}(1) = \boldsymbol{b}, \end{cases} \tag{H}$$

where $\Phi(\mathbf{u}) = (\varphi(u_1),\ldots,\varphi(u_n))$, $\varphi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism, $\boldsymbol{F} : (0,1) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. We notice that φ does not need to satisfy condition (A_1) in this section.

Definition 2.1. We say that $\boldsymbol{\alpha}$ is a *lower solution* of problem (H) if $\boldsymbol{\alpha} \in \prod_1^n C^2(0,1)$, $\Phi(\boldsymbol{\alpha}') \in \prod_1^n C^1(0,1)$ and

$$\begin{cases} \Phi(\boldsymbol{\alpha}'(t))' + \boldsymbol{F}(t,\boldsymbol{\alpha}) \ge \mathbf{0}, & t \in (0,1) \\ \boldsymbol{\alpha}(0) \le \boldsymbol{a}, \quad \boldsymbol{\alpha}(1) \le \boldsymbol{b}. \end{cases}$$