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q-Hardy–Berndt type sums associated with q-Genocchi type zeta and q-l-functions

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ABSTRACT

The aim of this paper is to define new generating functions. By applying a derivative operator and the Mellin transformation to these generating functions, we define q-analogue of the Genocchi zeta function, q-analogue Hurwitz type Genocchi zeta function, and q-Genocchi type l-function. We define partial zeta function. By using this function, we construct p-adic interpolation functions which interpolate generalized q-Genocchi numbers at negative integers. We also define p-adic meromorphic functions on $\mathbb{C}_p$. Furthermore, we construct new generating functions of q-Hardy–Berndt type sums and q-Hardy–Berndt type sums attached to Dirichlet character. We also give some new relations, related to these sums.

1. Introduction

In [34], we defined generating functions. By using these functions, we constructed q-Riemann zeta function, q-l-function and q-Dedekind type sum. This sum is defined by means of the following generating function:

$$\boldsymbol{Y}_b(h,k,q)=\sum_{m=1}^{\infty}\frac{\boldsymbol{f}\left(-\frac{2mi\pi h}{k},q\right)-\boldsymbol{f}\left(\frac{2mi\pi h}{k},q\right)}{m^b}$$

where h and k are coprime positive integers and b is an odd integer ≥ 1, and

$$\boldsymbol{f}(t,q)=\sum_{n=1}^{\infty}q^{-n}\exp\left(-q^{-n}[n]t\right),\text{ cf. [31,34]},\tag{1.1}$$

where q-number $[x]$ is defined by

$$[x]=[x:q]=\begin{cases}\dfrac{1-q^x}{1-q}, & (q\neq 1)\\ x, & (q=1)\end{cases}$$

cf. [8,14–24,35,36]; see also the references cited in each of these earlier works). If $q\in\mathbb{C}$, then we assume $|q|<1$.

In the remainder of our work, let $\exp(x)=e^x$ and χ be a Dirichlet character of conductor $f\in\mathbb{Z}^+$, the set of positive integer numbers.

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The so-called q-Dedekind type sums are given by the following theorem [34]:

Theorem 1. *Let h and k be positive integers and $(h,k)=1$ and assume that b is an odd integer ≥ 1. Then we have*

$$S_b(h,k;q)=\frac{b!}{(2\pi\mathrm{i})^b}\boldsymbol{Y}_b(h,k;q).$$

By using (1.1), we construct q-analogous of Hardy–Berndt sums $s_1(h,k)$ and $s_4(h,k)$. Our aim is to define generating functions of q-analogous of the Hardy–Berndt type sums $S(h,k)$, $s_2(h,k)$, $s_3(h,k)$ and $s_5(h,k)$. Therefore, we define the following generating function:

$$\mathcal{F}(t,q)=\sum_{n=1}^{\infty}(-1)^{n+1}q^{-n}\exp\left(-q^{-n}[n]t\right). \tag{1.2}$$

Applying the Mellin transformation to the Eq. (1.2), we construct, in Section 4, a family of q-Genocchi type zeta functions, which implies the classical Genocchi zeta functions. By applying the Mellin transformation to

$$\mathcal{F}(x,t,q):=\mathcal{F}(t,q)\exp(-tx),$$

we define a family of Hurwitz type q-Genocchi zeta functions.

Next, in terms of a Dirichlet character χ of conductor $f\in\mathbb{Z}^+$, we define a generalization of (1.2) by means of the following generating function:

$$\mathfrak{F}_\chi(t,q)=\sum_{n=1}^{\infty}(-1)^{n+1}\chi(n)q^{-n}\exp\left(-q^{-n}[n]t\right). \tag{1.3}$$

By applying the Mellin transformation to (1.3), we construct a family of q-Genocchi type l-functions.

By using (1.1)–(1.3), we prove our main results in Sections 7 and 8. By using (1.2), q-analogous of the Hardy–Berndt type sum (for example (2.1)) is defined by means of the following generating function:

$$Y_0(h,k;q)=\sum_{m=1}^{\infty}\frac{\mathcal{F}\left(-\frac{(2m-1)\mathrm{i}\pi h}{2k},q\right)-\mathcal{F}\left(\frac{(2m-1)\mathrm{i}\pi h}{2k},q\right)}{2m-1} \tag{1.4}$$

where h and k are coprime positive integers with $k\geq 1$.

Theorem 2. *Let h and k denote relatively prime integers with $k\geq 1$. If $h+k$ is odd, then*

$$S(h,k;q)=\frac{4}{\pi\mathrm{i}}Y_0(h,k;q). \tag{1.5}$$

Theorem 2 implies the classical Hardy–Berndt sums $S(h,k)$, which is given in (2.1). The other theorems, which are related to q-Hardy–Berndt type sums, $s_j(h,k;q)$, $j=1,2,3,4,5$, are given in Sections 7 and 8.

We define the sum $Y_{0,\chi}(t,q)$ as follows:

$$Y_{0,\chi}(h,k;q)=\sum_{m=1}^{\infty}\frac{\mathfrak{F}_\chi\left(-\frac{(2m-1)\mathrm{i}\pi h}{2k},q\right)-\mathfrak{F}_\chi\left(\frac{(2m-1)\mathrm{i}\pi h}{2k},q\right)}{2m-1}, \tag{1.6}$$

where h and k are coprime positive integers. Therefore, in terms of a Dirichlet character χ of conductor f, generalized q-Hardy–Berndt type sum $S_\chi(h,k;q)$ is given by the following theorem:

Theorem 3. *Let h and k denote relatively prime integers with $k\geq 1$. If $h+k$ is odd, then*

$$S_\chi(h,k;q)=\frac{4}{\pi\mathrm{i}}Y_{0,\chi}(h,k;q). \tag{1.7}$$

2. Hardy–Berndt sums and the Genocchi numbers

The classical Dedekind sums $s(h,k)$ first arose in the transformation formulae of the logarithm of the Dedekind eta-function. Similarly, the Hardy–Berndt sums arose in the transformation formulae of the logarithm of the theta-functions, $\operatorname{Log}\vartheta_n(0,q)$, $n=2,3,4$ cf. [1,2,4,6,12,13,30,38].