Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis 69 (2008) 3671–3678
Nonlinear Analysis
www.elsevier.com/locate/na

Review

# Existence and multiplicity of solutions for a non-periodic Schrödinger equation

Fukun Zhao[a,b,*], Leiga Zhao[b], Yanheng Ding[b]

[a] *Department of Mathematics, Yunnan Normal University, Kunming 650092 Yunnan, PR China*
[b] *Institute of Mathematics, AMSS, CAS, Beijing 100080, PR China*

Received 14 May 2007; accepted 12 October 2007

**Abstract**

This paper is concerned with the non-periodic Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

Under a general spectral assumption, the existence and multiplicity of solutions are obtained for asymptotically linear nonlinearity via variational methods.


*MR Subject Classification:* 37J45; 58E05; 34C37; 70H05

*Keywords:* Nonlinear Schrödinger equations; Variational methods; Asymptotically linear nonlinearities; Strongly indefinite functionals

**Contents**



* Corresponding author at: Department of Mathematics, Yunnan Normal University, Kunming 650092 Yunnan, PR China. Tel.: +86 8715516070; fax: +86 8715516070.
*E-mail address:* fukunzhao@163.com (F. Zhao).

0362-546X/$ - see front matter 
doi:10.1016/j.na.2007.10.024

## 1. Introduction and main results

In this paper we consider the existence of a nontrivial solution as well as multiplicity results for the semilinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^N \\ u(x) \to 0 \text{ as } |x| \to \infty. \end{cases} \tag{NS}$$

For the potential $V$, we assume

($V_1$) $V \in L^q_{loc}(\mathbb{R}^N, \mathbb{R})$, and $V^- := \min\{V, 0\} \in L^\infty(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ for some $q \in [2, \infty) \cap (N/2, \infty)$.

It is known that the assumption ($V_1$) ensures that the Schrödinger operator $S := -\Delta + V$ is self-adjoint and semibounded on $L^2(\mathbb{R}^N)$ (see Theorem A.2.7 in Simon [18]). We denote by $\sigma(S)$, $\sigma_{ess}(S)$, $\sigma_{pp}(S)$ the spectrum of $S$, the essential spectrum of $S$ and the pure point spectrum of $S$, respectively.

There is a large literature on existence and multiplicity results. It is not our purpose to give a survey in this paper. We only mention some related works here. In the case of $\inf \sigma(S) > 0$, there are many papers adopting various assumptions on $V$ and $f$; see for example [2,6,8,11,14,19,21,23] and references therein. In these papers solutions of (NS) were found as critical points of a functional which has the mountain pass geometry. For the case where 0 is a boundary point of a gap of $\sigma(S)$, Bartsch and Ding [5] obtained the existence and multiplicity results, which were extended later in Willem and Zou [22].

For the case where 0 lies in a gap of $\sigma(S)$, i.e.

$$\sup(\sigma(S) \cap (-\infty, 0)) < 0 < \inf(\sigma(S) \cap (0, \infty)),$$

Eq. (NS) has received growing attention in recent years. In [3] and [13], the primitive of $f(x,u)$ is strictly convex. Without the convexity condition, in [15] and [20], the authors obtained one nontrivial solution by establishing a new degree theory. In this case the variational functional is indefinite and even strongly indefinite (i.e. $\sigma_{ess}(S) \cap (-\infty, 0] \neq \emptyset$), so it has then a finite dimensional linking structure or an infinite dimensional linking structure (cf. Kryszewski and Szulkin [15]). The associated quadratic form $\int_{\mathbb{R}^N} (|\nabla u|2 + V(x)u^2) \mathrm{d}x$ is indefinite in sign, and this will make verification of compactness conditions a more delicate problem.

Since $H^1(\mathbb{R}^N) \hookrightarrow L^P(\mathbb{R}^N)$ is not compact, most of these papers consider the periodic (asymptotically periodic) or the radially symmetric problems. For the former, the periodicity is used to control the lack of compactness; one can make use of the invariance of the functional under translations to construct multi-bump solutions (see e.g. [1–3,5,8,10,11,15,16]). For the latter, one can work on the radially symmetric function space $H^1_r(\mathbb{R}^N)$ which possesses compactness of imbedding (see e.g. [19]). There are several authors who have dealt with non-periodic and non-radially symmetric problems. Liu, Su and Weth [17], via a mountain pass argument, obtained a positive solution, a negative solution and a sign-changing solution for asymptotically linear nonlinearity. Costa and Tehrani [7] obtained the existence and multiplicity results for the resonance case. Very recently, Ding and Jeanjean [9] has considered a class of the first order non-periodic Hamiltonian system.

In this paper we consider the non-periodic and non-radial problems. We assume

($V_2$) $a := \inf[(0, \infty) \cap \sigma_{ess}(S)] > 0$ and $b := \sup[(-\infty, 0) \cap \sigma_{ess}(S)] < 0$.

This implies that 0 is at most an eigenvalue of finite multiplicity of $S$. Moreover, ($V_2$) induces an orthogonal decomposition

$$L^2 = L^- \oplus L^+ \oplus L^0, \qquad u = u^+ + u^- + u^0$$

corresponding to the spectrum of $S$ such that $S$ is negative definite on $L^-$ and positive definite on $L^+$ and $L^0 = \ker S$. Denoting by $|S|$ the absolute value of $S$, let $E = D(|S|^{\frac{1}{2}})$ be the Hilbert space with the inner product

$$(u, v) = (|S|^{\frac{1}{2}} u, |S|^{\frac{1}{2}} v)_{L^2} + (u^0, v^0)_{L^2}$$

and norm $\|u\| = (u, u)^{\frac{1}{2}}$. We have a decomposition

$$E = E^- \oplus E^0 \oplus E^+, \quad \text{where } E^\pm = E \cap L^\pm \quad \text{and} \quad E^0 = L^0$$