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Existence of quasiperiodic solutions for the second-order approximation Boussinesq system

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Abstract

In this paper we study analytically a class of waves in the variant of the classical second-order approximation Boussinesq system given by

$$\partial_t u - b\partial_{xxt}u + c\partial_{xxxxt}u = -\partial_x v - \partial_x(uv) - \left(\frac{1}{3} - 2b\right)\partial_{xxx}v + b\partial_{xxx}(uv) + b\partial_x(u\partial_{xx}v) - a\partial_{xxxxx}v,$$

$$\partial_t v - b\partial_{xxt}v + c\partial_{xxxxt}v = -\partial_x u + \frac{1}{2}\partial_{xxx}u - v\partial_x v - \partial_x(u\partial_{xx}u) + b\partial_x(v\partial_{xx}v) - d\partial_{xxxxx}u,$$

where a, b, c, d are some real constants. This equation is ill-posed and most initial conditions do not lead to solutions. Nevertheless, we show that, for almost every a, b, c and d it admits solutions that are quasiperiodic in time.

MSC: primary 35Qxx; 37Kxx

Keywords: Boussinesq system; Normal forms; Hamiltonian formalism

1. Introduction

We study two-directional waves on the surface of an inviscid fluid in a flat channel without excluding the effects of wave interactions and/or wave reflections. In this case, a restricted four-parameter family of systems (see [2]),

$$\begin{aligned}
\partial_t u - B\partial_{xxt}u + B_1\partial_{xxxxt}u &= -\partial_x v - \partial_x(uv) - A\partial_{xxx}v + B\partial_{xxx}(uv) \\
&\quad - \left(A + B - \frac{1}{3}\right)\partial_x(u\partial_{xx}v) - A_1\partial_{xxxxx}v \\
\partial_t v - D\partial_{xxt}v + D_1\partial_{xxxxt}v &= -\partial_x u - C\partial_{xxx}u - v\partial_x v - C\partial_{xx}(v\partial_x v) - \partial_x(u\partial_{xx}u) \\
&\quad + (C + D - 1)\partial_x v\partial_{xx}v + (C + D)v\partial_{xxx}v - C_1\partial_{xxxxx}u
\end{aligned} \tag{1}$$

may be used. This system represents the second-order approximations to the Euler equations that govern the waves on the surface on an ideal fluid under the force of gravity. The dimensionless variables $u(x, t)$, $v(x, t)$, x and t are scaled

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by the length scale h_0 and time scale $(h_0/g)^{1/2}$ where h_0 denotes the still water depth and g denotes the acceleration of gravity. The variable $u(x, t)$ is the non-dimensional deviation of the water surface from its undisturbed position and $v(x, t)$ is the non-dimensional horizontal velocity at a height above the bottom of the channel corresponding to θh_0 with $0 \le \theta \le 1$. The constants A, B, C, D are called dispersive constants which satisfy the physical constraints

$$A + B + C + D = \frac{1}{3} \quad \text{and} \quad C + D = \frac{1-\theta^2}{2} \ge 0.$$

It is shown in [1,3] that this system has the capacity to capture the main characteristics of the flow in an ideal fluid.

In this paper, we will study system (1) when $A = \frac{1}{3} - 2D$, $C = -\frac{1}{2}$, $D = B := b$, $D_1 = B_1 := c$, $A_1 := a$, $C_1 = d$. Note that in this case Eq. (1) becomes

$$\begin{aligned} &\partial_t u - b\partial_{xxt} u + c\partial_{xxxxt} u = -\partial_x v - \partial_x(uv) - \left(\frac{1}{3} - 2b\right)\partial_{xxx} v + b\partial_{xxx}(uv) + b\partial_x(u\partial_{xx} v) - a\partial_{xxxxx} v, \\ &\partial_t v - b\partial_{xxt} v + c\partial_{xxxxt} v = -\partial_x u + \frac{1}{2}\partial_{xxx} u - v\partial_x v - \partial_x(u\partial_{xx} u) + b\partial_x(v\partial_{xx} v) - d\partial_{xxxxx} u, \end{aligned} \tag{2}$$

where a, b, c, d are real constants and where we have used that

$$\frac{1}{2}\partial_{xx}(v\partial_x v) + \left(b - \frac{3}{2}\right)\partial_x v\partial_{xx} v + \left(b - \frac{1}{2}\right) v\partial_{xxx} v = b\partial_x(v\partial_{xx} v).$$

We set

$$w_{1,k}(a, b) = 1 - \left(\frac{1}{3} - 2b\right)k^2 + ak^4, \qquad w_{2,k}(d) = 1 + \frac{1}{2}k^2 + dk^4 \tag{3}$$

and we restrict the values of a, b, c and d to the set $\mathcal{S}_1$ defined by

$$\mathcal{S}_1 = \left\{(a, b, c, d) \in \mathbb{R}^4 : w_{1,k}(a, b)w_{2,k}(d) < 0 \ \text{ for } k \in \{1, 2\}\right\}.$$

We note that we could have also worked with the values of the parameters $A, B, C, D, A_1, B_1, C_1, D_1$ satisfying

$$\frac{(1 - Ak^2 + A_1k^4)(1 - Ck^2 + C_1k^4)}{(1 + Dk^2 + D_1k^4)^2} < 0 \tag{4}$$

for some $k \in \{1, \dots, N\}$ and some finite integer $N \ge 2$. For any other value of the parameters in (1) not satisfying Eq. (4), additional techniques may be used in order to prove the existence of quasiperiodic solutions for system equation (2). However, due to tedious calculations in the computation of the Hamiltonian, of the normal form and in the verification of the nondegeneracy conditions for the KAM theory, we restrict to the values of $A, B, C, D, A_1, B_1, C_1, D_1$ explained above with $a, b, c, d \in \mathcal{S}_1$.

The present paper deals with the existence of periodic and quasiperiodic solutions (with two frequencies) for system Eq. (2) that generalize the linear oscillations of the normal flow to the complete system. These solutions are obtained inside a certain invariant center manifold for the dynamics, which turns out to be *finite-dimensional*. The approach of reducing to a center manifold can be traced back to the work of Kirchgässner [7], and the method is sometimes called Kirchgässner reduction. The application of this approach strongly depends on the desired results and appropriate additional techniques may need to be developed in each particular case, mainly due to the infinite-dimensional nature of the problems. For later developments and applications of this method we refer the reader to the works [4–6,8–10,12] and to the references therein.

Once we have proven the existence of a center manifold, we will study the dynamics of the restriction to this center manifold. To do it, first we observe that system (2) can be written as a Hamiltonian in infinitely many coordinates, which turns out to be real analytic near the origin. Then we will compute the Birkhoff normal form up to fourth order of the Hamiltonian restricted to the center manifold. This will allow us to check that the nondegeneracy conditions for the KAM theorem are satisfied and thus by means of KAM techniques to prove the existence of quasiperiodic solutions for this transformed Hamiltonian on the center manifold.

The paper has been organized in the following way: in Section 2 we provide the main result of this paper and its proof by stating some auxiliary results like the existence of finite center manifolds, the computation of the Hamiltonian