Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis 69 (2008) 3866–3876
Nonlinear Analysis
www.elsevier.com/locate/na

Periodic solutions for second-order differential equations with a singular nonlinearity☆

Xiong Li*, Ziheng Zhang

School of Mathematical Sciences, Beijing Normal University, Beijing 100875, People's Republic of China

Received 9 July 2007; accepted 12 October 2007

Abstract

This paper deals with the existence of positive T-periodic solutions for the following differential equation

$$\ddot{x} + a(t)x = f(t, x) + c(t),$$

where $a, c \in L^1[0, T]$ and $f \in \mathrm{Car}([0, T] \times \mathbb{R}^+, \mathbb{R})$. The existence results are obtained by using a fixed point theorem in cones.

MSC: 34B16; 34B18; 34B27

Keywords: Positive periodic solutions; Singularity; Green's function

1. Introduction

The main motivation of this paper is the search of non-trivial positive T-periodic solutions for second-order differential equations like

$$\ddot{x} + a(t)x = \frac{b(t)}{x^\alpha} + c(t) \tag{1.1}$$

with $a, b, c \in L^1[0, T]$ and $\alpha > 0$.

In the recent years, the periodic problem for the semilinear singular equation (1.1) has deserved the attention of many specialists in differential equations. The interest in scalar equations with singularity began with some works of Forbat and Huaux [7,14], where the singular nonlinearity models the restoring force caused by a compressed perfect gas (see [17] for a more complete list of references). Later, the interest in this problem increased with the paper of Lazer and Solimini [16]. They proved that, for $a(t) \equiv 0$, $b(t) \equiv 1$, $\alpha \geq 1$ (called the strong force condition in a terminology first introduced by Gordon [11,12]), a necessary and sufficient condition for the existence of a positive

☆ This project was supported by the Program for New Century Excellent Talents of Ministry of Education of China and the National Natural Science Foundation of China (Grant Nos. 10671020 and 10301006).
* Corresponding author.
E-mail address: xli@bnu.edu.cn (X. Li).

0362-546X/$ - see front matter
doi:10.1016/j.na.2007.10.023

T-periodic solution is that the mean value of c is negative. Moreover, if $0 < \alpha < 1$, they only showed that they could find some functions c with negative mean values such that the following differential equation

$$\ddot{x} = \frac{b}{x^{\alpha}} + c(t) \tag{1.2}$$

with $0 < \alpha < 1$, $b > 0$ and $c(t) \in L^1[0, T]$, does not have any T-periodic solution. This work is a hallmark for the problem and since its publication many researchers have focused their attention on the study of singular equations.

Since then, the strong force condition ($\alpha \geq 1$) became standard in the related works, see for instance [1,2,4,6,8,9,13,15,18–20,23,24,27,30,31], the recent review [21] and their bibliographies. With the strong singularity, the energy near the origin becomes infinity and this fact is helpful for obtaining either a priori bounds needed for the application of the degree theory, or the fast rotation needed in a recent version of the Poincaré–Birkhoff Theorem. Here, we must mention the result in [19], it is proved that the following differential equation

$$\ddot{x} + \mu x = \frac{b}{x^{\alpha}} + p(t) \tag{1.3}$$

possesses a T-periodic solution for $\alpha \geq 1$, $b > 0$, $p(t) \in L^1[0, T]$ and $\mu \neq (\frac{k\pi}{T})^2$ for all $k \in \mathbb{Z}$, and moreover, the open problem of finding additional conditions on p to ensure the existence of T-periodic solutions in the cases $\mu = (\frac{k\pi}{T})^2$ for $k \in \mathbb{Z}$, i.e., the resonant case is explicitly quoted. From this point of view, Lazer and Solimini's results in [16] correspond to some conditions on p to deal with the resonant case $\mu = 0$ in Eq. (1.3).

In [20], for the first time, the authors proved that if $\mu = 0$ (the zero resonant point), Eq. (1.3) has at least one positive T-periodic solution, assume that the mean value of p is negative and $p(t)$ has a uniform lower bound; if $\mu = (\frac{\pi}{T})^2$ (the first resonant point), Eq. (1.3) has at least one positive T-periodic solution, under the condition that $p(t)$ is positive, which does not require the strong force assumption $\alpha \geq 1$, using the method of lower and upper solutions. These conclusions had been improved in the more recent paper [1].

In Section 4, we will deal with the existence of positive T-periodic solution of the differential equation like Eq. (1.2), using a fixed point theorem in cones, which is a common technique to approach the periodic boundary value problem.

On the other hand, if compared with the literature available for the strong singularity, the study of the existence of periodic solutions under the presence of the weak singularity ($0 < \alpha < 1$) is much more recent and the number of references is considerably smaller, see for instance [3,10,20,25,26].

Form now on, we denote by e_* and e^* the essential supremum and infimum of a given function $e \in L^1[0, T]$, if they exist. In the above mentioned paper [26], the author proved that under the assumptions (H1) and (H2) (see Section 3 for details) with $0 < \alpha < 1$, the differential equation

$$\ddot{x} + a(t)x = f(t, x) + c(t) \tag{1.4}$$

has a positive T-periodic solution, if $\gamma^* \leq 0$ and

$$\gamma_* \geq \left[\frac{\hat{\beta}_*}{(\beta^*)^{\alpha}}\alpha^2\right]^{\frac{1}{1-\alpha^2}}\left(1 - \frac{1}{\alpha^2}\right), \tag{1.5}$$

where $\gamma(t)$, $\beta(t)$ and $\hat{\beta}(t)$ are defined in Section 3, which is a uniform lower bound on γ_*.

In Section 3, we still discuss the existence of positive T-periodic solutions of Eq. (1.4) under the assumptions (H1) and (H2) with $\alpha > 0$ and will prove that either $c_* \geq 0$ or $c_* < 0$ and

$$\gamma^* \leq \left(\frac{\hat{b}_*}{|c_*|}\right)^{1/\alpha} + \frac{\beta^* c_*}{\sigma^{\alpha}\hat{b}_*}, \tag{1.6}$$

where σ and $\hat{b}(t)$ are defined in Sections 2 and 3 respectively, Eq. (1.4) has a positive T-periodic solution. Contrary to the above mentioned result, we do not restrict ourselves to the weak singularity, and such a conclusion is also valid under the strong force condition. On the other hand, the assumption (1.6) which is a uniform upper bound on γ^*