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ScienceDirect
Nonlinear Analysis 69 (2008) 4025–4054
Nonlinear Analysis
www.elsevier.com/locate/na

Semigroups of locally Lipschitz operators associated with semilinear evolution equations of parabolic type

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Received 22 May 2007; accepted 18 October 2007

Abstract

A characterization problem is discussed, of semigroups of locally Lipschitz operators providing mild solutions to the Cauchy problem for the semilinear evolution equation of parabolic type $u'(t) = (A + B)u(t)$ for $t > 0$. By parabolic type we mean that the operator A is the infinitesimal generator of an analytic (C_0) semigroup on a general Banach space X. The operator B is assumed to be locally continuous from a subset of Y into X, where Y is a Banach space which is contained in X and has a stronger norm defined through a fractional power of $-A$. The characterization is applied to the global solvability of the mixed problem for the complex Ginzburg–Landau equation.

MSC: 47H14; 47H20; 34G20

Keywords: Semigroup of locally Lipschitz operators; Semilinear evolution equation of parabolic type; Analytic semigroup; Fractional power; Mild solution; Smoothing effect

1. Introduction

Let X be a general Banach space with norm $\|\cdot\|$ and D a subset of X. By a *semigroup on D* we mean a one-parameter family $\{S(t); t \geq 0\}$ of operators from D into itself satisfying the so-called *semigroup property* and the strong continuity in $t \geq 0$. In order to develop a general theory of nonlinear semigroups, it is necessary to consider the continuity of the operators $S(t)$ in an appropriate way. In this paper, a vector-valued functional $\varphi = (\varphi_i)_{i=1}^n$, satisfying that $\varphi_i : X \to [0, \infty]$ and the effective domain $D(\varphi_i)$ contains the set D for $1 \leq i \leq n$, is employed to consider the continuity condition of the operators $S(t)$ in such a way that for $\tau \geq 0$ and $r \in \mathbb{R}_+^n$ there exists $L_{\tau,r} > 0$ satisfying that

$$\|S(t)x - S(t)y\| \leq L_{\tau,r}\|x - y\| \quad \text{for } x, y \in D_r \text{ and } t \in [0, \tau],$$

where $\mathbb{R}_+ = [0, \infty)$ and $D_r = \{x \in D; \varphi(x) \leq r\}$ for $r \in \mathbb{R}_+^n$. Here vectorial inequalities are used on the understanding that the same inequalities hold between their respective components. A semigroup on D satisfying

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0362-546X/$ - see front matter
doi:10.1016/j.na.2007.10.035

the above-mentioned continuity condition is called a *semigroup of locally Lipschitz operators on D with respect to the functional* φ. The generation of semigroups of locally Lipschitz operators has been recently studied in several settings. Among others, a characterization of nonlinearly perturbed (C_0) semigroups was given in [9] and applied to the global solvability of the mixed problem for the complex Ginzburg–Landau equation

$$\partial u/\partial t - (\lambda + \mathrm{i}\mu)\Delta u + (\kappa + \mathrm{i}\nu)|u|^{q-2}u - \gamma u = 0 \quad \text{in } \Omega \times (0, \infty),$$

where $2 \leq q \leq 2 + 4/N$ and Ω is a general domain of $\mathbb{R}^N$ with dimensional restriction $1 \leq N \leq 4$. In this connection, it should be noticed that by establishing an abstract theory involved with subdifferential operators, Okazawa and Yokota first showed in [19] that the solution operators associated with the mixed problem for the complex Ginzburg–Landau equation in a bounded domain form a semigroup of locally Lipschitz operators on $L^2(\Omega)$ and have a smoothing effect in the sense that the solution $u(t)$ with initial data $u_0 \in L^2(\Omega)$ belongs to the space $H^2(\Omega) \cap H_0^1(\Omega)$ for almost all $t > 0$. After that, in [24] they succeeded in proving that the same result holds for a general domain Ω of $\mathbb{R}^N$ without dimensional restriction.

This paper is motivated by the abstract results in [19,24] concerning semigroups of locally Lipschitz operators having smoothing effects. We are interested in studying a characterization of nonlinearly perturbed analytic (C_0) semigroups. In order to characterize nonlinearly perturbed analytic (C_0) semigroups, we interpret such a problem as a characterization problem, of semigroups of locally Lipschitz operators providing mild solutions to the Cauchy problem for the semilinear evolution equation of parabolic type

$$\text{(SP)} \quad u'(t) = (A + B)u(t) \quad \text{for } t > 0.$$

By *parabolic type* we mean that the operator A is the infinitesimal generator of an analytic (C_0) semigroup $\{T(t); t \geq 0\}$ on X. The operator B is assumed to be locally continuous from the subset $D \cap Y$ of Y into X, where Y is a Banach space which is contained in X and has a stronger norm defined through a fractional power of $-A$. By using the above-mentioned functional φ, a local continuity condition on B will be given in a spirit similar to that a family of restrictions of a given operator, by the use of a well-chosen functional, was formed in [23], such that each restriction is dissipative and satisfies the so-called range condition, although the given operator is not dissipative. This idea goes back at least as far as [3].

The semilinear problem (SP) has been studied by many authors. If B is locally Lipschitz continuous from the set $D \cap Y$ into X, then the local solvability for (SP) can be shown in [1,14] by the Banach–Picard fixed point theorem. In the setting where B is locally continuous from the set $D \cap Y$ into X, the construction of approximate solutions was done under subtangential conditions of various types. In [13] the subtangential condition $\liminf_{h\downarrow 0} h^{-1}d(v + hBv, D) = 0$ for $v \in D \cap Y$ was used under the assumption that D is invariant under $T(t)$. In [2] the time-dependent case was considered in a similar setting, and the condition $\lim_{h\downarrow 0} h^{-1}d(T(h)v + hBv, D) = 0$ for $v \in D \cap Y$ was used in [4]. Prüss proposed in [21] the following subtangential condition: There exists $\eta > 0$ such that to each $v \in D \cap Y$ and $\varepsilon > 0$ there correspond $h > 0$ and $w_h \in D \cap Y$, and z_h defined by

$$z_h = T(h)v + \int_0^h T(\xi)Bv\mathrm{d}\xi - w_h$$

satisfying $\|z_h\| \leq \varepsilon h$ and $\|(-A)^\alpha z_h\| \leq \varepsilon h^\eta$. This condition is necessary for the existence of local mild solutions. The choice of this condition seems to be the best adapted to our theory.

In the above-mentioned papers, the compactness of the resolvent of the operator A, the locally Lipschitz continuity on B, or the quasi-dissipativity of $A + B$ was assumed to show the convergence of a sequence of the constructed approximate solutions. The main theorem (Theorem 5.2) provides a natural and intrinsic characterization of nonlinear perturbed analytic (C_0) semigroups. We emphasize that there exists a family of metric-like functionals by means of which the semilinear stability condition (ii-2) in Theorem 5.2 is necessary for local mild solutions depending Lipschitz continuously on their initial data to exist and that the semilinear stability condition introduced here is used to prove the convergence of a sequence of approximate solutions. This means that the semilinear stability condition together with the subtangential condition in the sense of Prüss is necessary and sufficient for generating a semigroup of locally Lipschitz operators providing mild solutions to (SP).

This paper is organized as follows. In Section 2 we impose basic assumptions on A and B appearing in (SP) and give the uniqueness and regularity of mild solutions. The Appendix is devoted to the definition and properties