Nonlinear Analysis 72 (2010) 77–98

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Global solutions and asymptotic behavior for a parabolic degenerate coupled system arising from biology

Gabriela Liţcanu [a,b,*], Cristian Morales-Rodrigo [c]

[a] Institute of Mathematics "O. Mayer", Romanian Academy, 700505 Iaşi, Romania
[b] Institute of Mathematics, Witten/Herdecke University, D-58453 Witten, Germany
[c] Institute of Applied Mathematics and Mechanics, Warsaw University, 02-097 Warsaw, Poland

ARTICLE INFO

Article history:
Received 17 January 2009
Accepted 12 June 2009

MSC:
35B30
35B40
35B45
92C17

Keywords:
Parabolic-degenerate system
Global existence
Classical solutions
Steady state
Asymptotic behavior

ABSTRACT

In this paper we will focus on a parabolic degenerate system with respect to unknown functions u and w on a bounded domain of the two dimensional Euclidean space. This system appears as a mathematical model for some biological processes. Global existence and uniqueness of a nonnegative classical Hölder continuous solution are proved. The last part of the paper is devoted to the study of the asymptotic behavior of the solutions.

1. Introduction

During the last years models originated from biology earned a privileged place in mathematical modeling and became the focus of interest of mathematicians and biologists as well. In many cases the study of these models involves challenging mathematical problems that originate in the intrinsic mathematical structure of the model. Moreover the possibility of taking suitable hypotheses is limited by the necessity to fit with experimental data of the process the model originates in.

Let us consider the following initial-boundary problem:

$$\frac{\partial u}{\partial t} = a\Delta u - b\nabla\cdot(u\chi(w)\nabla w) + f(u,w) \quad x\in\Omega,\ t\in\mathbb{R}_+ \tag{1.1}$$

$$\frac{\partial w}{\partial t} = -kw^{\beta}u \quad x\in\Omega,\ t\in\mathbb{R}_+ \tag{1.2}$$

$$u(x,0) = u_0(x) \geqslant 0 \quad x\in\Omega \tag{1.3}$$

$$w(x,0) = w_0(x) > 0 \quad x\in\Omega \tag{1.4}$$

* Corresponding author at: Institute of Mathematics "O. Mayer", Romanian Academy, 700505 Iaşi, Romania. Tel.: +40 0 2 32 21 11 50; fax: +40 0 2 32 21 11 50.
E-mail addresses: litcanu@uaic.ro (G. Liţcanu), cristianmatematicas@yahoo.com (C. Morales-Rodrigo).

0362-546X/$ – see front matter
doi:10.1016/j.na.2009.06.083

where $\Omega \subseteq \mathbb{R}^N$ is a domain, a, b and k are positive constants, $\chi(w) = w^{-\alpha}$, $0 \leqslant \alpha < 1$, $\beta \geqslant 1$ and f is a given function. If Ω is bounded, then the system (1.1)–(1.4) is considered together with the no-flux boundary condition

$$\frac{\partial u}{\partial \eta} - u\chi(w)\frac{\partial w}{\partial \eta} = 0 \quad x \in \partial\Omega,\ t \in \mathbb{R}_+ \tag{1.5}$$

where η denotes the unit outward normal vector of $\partial\Omega$.

This system is a particular version of the well-known mathematical model proposed by Keller and Segel [1] (see also [2–4]) with an additional reaction term $f(u, w)$ in the first equation. The Keller–Segel model was proposed in order to describe the spatial aggregation of cellular slime molds which move toward high concentrations of some chemical substance secreted by the cells themselves. The function $u(x, t)$ describes the density distribution of the cell population, $w(x, t)$ denotes the concentration of the chemical substance at a position $x \in \Omega$ and a time $t \in \mathbb{R}_+$ and the function χ is the chemotactic sensitivity.

The classical Keller–Segel model, when the second variable is also supposed to be diffusive, has been the subject of many papers (see, for example, the surveys of Horstmann [5,6] and the references given therein). In the literature there are many theoretical results for the Keller–Segel model concerning existence and uniqueness as well as the qualitative behavior of the solutions. Most of the results were focused on the global existence of solutions versus blow-up in finite time. Both behaviors strongly depend on the initial data and space dimension.

The system (1.1)–(1.5) also appears as a simplified version of a mathematical model describing the tumor growth. In this case, the function $u(x, t)$ describes the tumor cells density and $w(x, t)$ denotes the density of the extracellular matrix (the surrounding healthy tissue degraded locally by the action of tumor cells). There are several models of different stages of this process incorporating also the action of some degradative enzymes, cell cycle elements or cell age structures. For a more thorough biological background and numerical results see, for example, [7–11]. We refer also to [12,13] where the global existence and uniqueness of solutions in the case of some systems related with this process are investigated.

Previously, a version of the system (1.1)–(1.5) was studied by Rascle in [14] (see also [15]) with the boundary condition (1.5) replaced with

$$\frac{\partial u}{\partial \eta} = 0. \tag{1.6}$$

Instead of (1.4) he takes a positive constant as initial condition for the function w, $w_0(x) \equiv w_0 > 0$ and $f(u, v)$ satisfying the following condition

$$\exists L > 0, \quad \forall u \in \mathbb{R}, \quad \forall w > 0, \qquad |f(u, w)| \leqslant L\,|u|\,. \tag{1.7}$$

In the previous hypotheses, the local existence and uniqueness of a classical Hölder continuous solution of the system (1.1)–(1.4) has been proved when $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. The global existence has been shown in one space dimension. We mention that another result, in the one dimensional space, concerning the global existence and uniqueness of classical solutions for a similar system is given in [16].

In more than one dimension, when Ω is the whole space $\mathbb{R}^N$, the system (1.1)–(1.5) has been considered in [17–19] with $\chi(w)$ a given positive function on $\mathbb{R}_+$ such that $w\chi(w)$ is strictly increasing (thus including the case $\chi(w) = w^{-\alpha}$, $0 \leqslant \alpha < 1$) and $f \equiv 0$. In this case the global existence of weak solutions has been proved.

In [20] the authors considered the problem (1.1)–(1.4) in a more general form under Dirichlet conditions. Assuming that a priori L^∞ bounds are available they proved the local and global existence of weak solutions.

Finally, we cite here the paper [21] where the author considers instead of Eq. (1.2) the following one

$$\frac{\partial w}{\partial t} = g(u, w)$$

but under some hypotheses on g that are not satisfied in the case we shall consider in this paper (see also [22,23]).

Our aim in this paper is to prove the global existence in time and uniqueness of a classical Hölder continuous solution for the problem (1.1)–(1.4) when $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$ and the reaction term is the logistic growing function. Also the long time asymptotic behavior of the solution is investigated. In order to simplify the presentation of the results, we shall consider in what follows the case $\alpha = 0$. The more general cases $\beta \geqslant 1$, $0 \leqslant \alpha < 1$ (or even when the function χ is a more general decreasing function) can be treated similarly, the estimations being more tedious.

This paper is organized as follows. In Section 2 we review some basic facts concerning the notations and terminology used through the paper and we also give some auxiliary results. The proof of the local existence in time and uniqueness of a classical solution is accomplished by applying a fixed point argument in a suitably chosen function space and is presented in Section 3.

In Section 4 we will be concerned with the global existence in time of the classical solutions and for this we will begin by establishing a priori bounds.

In Section 4.1 we obtain a Lyapunov function for the system (independent of the space dimension) by an analogous method as in [18] (see also [19,24,25]). We derive $L^p(\Omega)$ estimates independent of time in Section 4.2. After establishing a priori $L^\infty(\Omega)$ uniform bounds in Section 4.3, we proceed to prove the existence of global Hölder continuous solutions imposing that the initial data are smooth enough.