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A class of degenerate parabolic equations with a concentrated nonlinear source

Jingxue Yin, Shuhe Wang, Yuanyuan Ke *

Department of Mathematics, Jilin University, Changchun 130012, PR China

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ABSTRACT

In this paper, we consider the Cauchy problem for a class of degenerate parabolic equations with a concentrated nonlinear source. We obtain the existence of the generalized solutions for the problem based on some a priori estimates on solutions.

1. Introduction

In this paper, we study the following Cauchy problem for the degenerate parabolic equation with a concentrated nonlinear source

$$\frac{\partial u}{\partial t} - (u^m)_{xx} = \delta(x) f(u(x,t)), \quad (x,t) \in Q_T, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $\delta(x)$ is the Dirac measure, $m > 1$, $Q_T = \mathbb{R} \times (0,T)$.

Initially, mathematical workers considered the partial differential equations with Dirac measure $\delta(x)$ as the sources, see for example [1–4]. From then on, many authors focused their interests on the problems with the source of the form $\delta(x) f(u(x,t))$, such as [5–9]. Our consideration is motivated by the work of Olmstead and Roberts [5], where they studied the equation

$$\frac{\partial u}{\partial t} - u_{xx} = \delta(x-b) f(u(x,t))$$

with the boundary and initial value conditions. This model can be used to describe the temperature of a 1D strip of a finite width that contains a concentrated nonlinear source at b. The authors discussed the possibility of a blow-up solution of the problem by using Green's function and analyzing its corresponding nonlinear Volterra equation of the second kind. Moreover, Chan and Tian [7] considered the following equation

$$x^q \frac{\partial u}{\partial t} - u_{xx} = a^2 \delta(x-b) f(u(x,t)).$$

They proved that under some conditions, the solution blows up in a finite time, and they gave the range of the blow-up time. For the porous medium equation, when the nonlinear source term in Eq. (1.1) is replaced by the simple form $\delta(x)$, the

* Corresponding author.
E-mail address: keyy@jlu.edu.cn (Y. Ke).

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existence of BV solutions was obtained by Yuan and Jin [10] based on some BV estimates. And also there is a rich literature concerning these types of parabolic problems when the nonlinear source is of the form $f(u(x,t))$, such as [11], where the authors discuss the existence of local solutions by using the regularized methods.

In the present paper, the main purpose is to show the existence of the generalized solutions for the problem (1.1)–(1.2). Since $m > 1$, the appearance of the diffusion term makes the theory of Green's function used in [5,7] inapplicable. While due to the appearance of the concentrated source $\delta(x)$, Hopf's maximum principle could no longer be applied to derive the L^∞ estimate on solutions as in [11]. In addition, since the source is associated with the nonlinear function $f(u(x,t))$, some results such as the local boundedness of the solutions cannot be obtained by using the methods in [10]. In this paper, by utilizing some iteration technologies, we obtain the local boundedness of the solutions.

2. Main result and its proof

First, we present the assumptions and the definition of solutions.

(H1) $f \in C^1(\mathbb{R})$, and $f(s) > 0$, for $s > 0$.
(H2) $u_0(x) \in C^\alpha(\mathbb{R})$, $\alpha \in (0,1)$, $u_0(x) \geq 0$, and $\operatorname{supp} u_0 = \overline{\Omega}$, Ω is an open and bounded set in $\mathbb{R}$.

In this paper, our main efforts center on the discussion of generalized solutions, since the regularity follows from a quite standard approach. Hence we give the following definition of generalized solutions of the problem (1.1)–(1.2).

Definition 2.1. A nonnegative function $u : Q_T \to [0,+\infty)$ is called a generalized solution of problem (1.1)–(1.2) on Q_T, if $u \in L^\infty(0,T; L^\infty_{\mathrm{loc}}(\mathbb{R})) \cap C((\mathbb{R}\setminus\{0\}) \times (0,T))$, for any $\varphi(x,t) \in C_0^\infty(Q_T)$ and $\psi(x) \in C_0^\infty(\mathbb{R})$, it follows that

$$\iint_{Q_T} \left(-u\varphi_t - u^m\varphi_{xx}\right)\mathrm{d}x\mathrm{d}t = \int_0^T f(u(0,t))\varphi(0,t)\mathrm{d}t,$$

and

$$\lim_{t\to 0^+}\int_{\mathbb{R}} \psi(x)u(x,t)\mathrm{d}x = \int_{\mathbb{R}} \psi(x)u_0(x)\mathrm{d}x.$$

The main result of this paper is the following

Theorem 2.1. *Let* (H1)–(H2) *be valid. Then the Cauchy problem* (1.1)–(1.2) *admits a generalized solution on* Q_T.

In order to study the existence of the generalized solutions, we should first consider the following approximate problem

$$\frac{\partial u_\varepsilon}{\partial t} - \frac{\partial^2(u_\varepsilon^m)}{\partial x^2} = \delta_\varepsilon(x)f(u_\varepsilon(x,t)), \quad (x,t) \in Q_T, \tag{2.1}$$

$$u_\varepsilon(x,0) = u_0(x) + \varepsilon, \quad x \in \mathbb{R}, \tag{2.2}$$

where

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon}j\left(\frac{x}{\varepsilon}\right), \quad 0 < \varepsilon < 1,$$

$$j(x) = \begin{cases} \dfrac{1}{A}\mathrm{e}^{1/(|x|^2-1)}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

$$A = \int_{-1}^{1} \mathrm{e}^{1/(|x|^2-1)}\mathrm{d}x.$$

It is obvious that

$$\int_{\mathbb{R}} j(x)\mathrm{d}x = 1, \quad 0 \leq \delta_\varepsilon(x) \in C_0^\infty(\mathbb{R}),$$

$$\int_{\mathbb{R}} \delta_\varepsilon(x)\mathrm{d}x = 1, \quad \operatorname{supp}\delta_\varepsilon(x) = \{x \in \mathbb{R}, |x| \leq \varepsilon\},$$

and hence we have

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} \delta_\varepsilon(x)\phi(x)\mathrm{d}x = \phi(0), \quad \forall \phi \in C(\mathbb{R}).$$

By the classical theories of the parabolic equations, the problem (2.1)–(2.2) admits a unique nonnegative solution u_ε.

Next we do the a priori estimates of u_ε.