Nonlinear Analysis 72 (2010) 151–163

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On a class of second order differential inclusions driven by the scalar p-Laplacian

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ARTICLE INFO

Article history:
Received 11 October 2008
Accepted 12 June 2009

MSC:
34K30
47D60
49K25

Keywords:
Sobolev space
Ordered Banach space
Upper and lower solutions
Upper semicontinuous map
Completely continuous operator
Fixed point
Maximal monotone map
Yosida approximation

ABSTRACT

In this paper, we study a class of nonlinear value boundary problems for second order differential inclusions. By using the methods of upper–lower solutions, truncations and penalization, we give some existence results of the extremal (greatest and smallest) solutions. Our approach is based on the theories of fixed points, maximal monotone operators and ordered Banach spaces.

1. Introduction and preliminaries

In the last decade, the second order differential systems driven by p-Laplacian operators (or p-Laplacian-like operators) have attracted increasing interest. Many works were carried out with various techniques employed, such as Pino–Elgueta–Manásevich [1], Manásevich–Mawhin [2], Zhang [3], Aizicovici–Papageorgiou–Staicu [4] with the Leray–Schauder degree, Kyritsi–Matzakos–Papageorgiou [5], Panalini [6], Zhang–Li [7] with fixed points of the multivalued maps, Bader–Papageorgiou [8], Papageorgiou–Staicu [9] with the method of upper–lower solutions etc.

Motivated by [8,9], we deal with the second order differential inclusion with nonlinear boundary values:

$$\begin{cases} (|x'(t)|^{p-2}x'(t))' \in Ax(t) + F(t, x(t), x'(t)) + \theta(x(t)) & \text{a.e. on } T = [0, b], \\ x'(0) \in \xi_1(x(0)), -x'(b) \in \xi_2(x(b)), \end{cases} \tag{1.1}$$

where $A : \mathscr{D}(A) \subseteq \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}, \xi_i : \mathscr{D}(\xi_i) \subseteq \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}, i = 1, 2$ are maximal monotone maps and $F : T \times \mathbb{R}^2 \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is a multivalued map upper semicontinuous about the second variable ($2 \le p < +\infty$).

As in many cases, the map A is not assumed to be bounded; with the domain $\mathscr{D}(A)$ no longer the whole $\mathbb{R}$, it is difficult to give a suitable condition like (H_F) (iii) in [9], or $H(8)$ in [10], which is used to guarantee the boundedness for the derivatives of any potential solutions of problem (1.1). Instead, we strengthen the hypothesis (H_F) (iv) in [9] as the linear growth condition.

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doi:10.1016/j.na.2009.06.041

For a.e. $t \in T$, $\forall\, x, y \in \mathbb{R}$ and $\forall\, z \in F(t, x, y)$, we have

$$|z| \leq \gamma_1(t, |x|) + \gamma_2(t, |x|)|y|,$$

and

$$\sup_{r\leq k} \gamma_1(t, r) \leq \eta_{1,k}(t), \quad \sup_{r\leq k} \gamma_2(t, r) \leq \eta_{2,k}(t),$$

where $\eta_{1,k} \in L^2(T, \mathbb{R}^+)$, and $\eta_{2,k} \in L^{2p/p-2}(T, \mathbb{R}^+)$ (or $L^\infty(T, \mathbb{R}^+)$ if $p = 2$).

By employing the methods of upper–lower solutions, truncations and penalization, we investigate the existence of extremal (greatest and smallest) solutions of (1.1). Our approach relies on the theories of fixed points, maximal monotone operators and ordered Banach spaces. The final results in this work, which give a positive answer to the question raised at the end of [9], can be viewed as the complements and extensions to the corresponding conclusions in [8,9].

It is worth pointing out that the upper–lower solutions, as well as the conditions of Hartman's type, are special cases of the solution tubes (see [7,10,11]).

Firstly, we give some definitions associated with (1.1).

By a solution of (1.1), we mean a function $x \in C^1(T)$ with $|x(\cdot)|^{p-2}x(\cdot) \in W^{1,q}(T)(\frac{1}{p} + \frac{1}{q} = 1)$, satisfying $x(t) \in \mathscr{D}(A)$, and

$$(|x(t)|^{p-2}x(t))' = h(t) + f(t) + \theta(x(t)) \quad \text{a.e. } t \in T, \tag{1.2}$$

with the boundary conditions holding. Here $h, f \in L^q(T)$ satisfying $h(t) \in Ax(t)$ and $f(t) \in F(t, x(t), x'(t))$ a.e. on T.

By a lower solution of (1.1), we mean a function $\psi \in C^1(T)$ with $|\psi(\cdot)|^{p-2}\psi(\cdot) \in W^{1,q}(T)$ satisfying $\psi(t) \in \mathscr{D}(A)$ a.e. on T, and

$$\begin{cases} (|\psi'(t)|^{p-2}\psi'(t))' \geq g_\psi(t) + f_\psi(t) + \theta(\psi(t)) & \text{a.e. } t \in T, \\ \psi'(0) \in \xi_1(\psi(0)) + \mathbb{R}^+, \qquad -\psi'(b) \in \xi_2(\psi(b)) + \mathbb{R}^+, \end{cases} \tag{1.3}$$

where $g_\psi \in S^q_{A\psi(\cdot)}, f_\psi \in S^q_{F(\cdot,\psi(\cdot),\psi'(\cdot))}$.

Similarly, by an upper solution of (1.1), we mean a function $\varphi \in C^1(T)$ with $|\varphi(\cdot)|^{p-2}\varphi(\cdot) \in W^{1,q}(T)$ satisfying $\varphi(t) \in \mathscr{D}(A)$ a.e. on T, and

$$\begin{cases} (|\varphi'(t)|^{p-2}\varphi'(t))' \leq g_\varphi(t) + f_\varphi(t) + \theta(\varphi(t)) & \text{a.e. } t \in T, \\ \varphi'(0) \in \xi_1(\varphi(0)) - \mathbb{R}^+, \qquad -\varphi'(b) \in \xi_2(\varphi(b)) - \mathbb{R}^+, \end{cases} \tag{1.4}$$

where $g_\varphi \in S^q_{A\varphi(\cdot)}, f_\varphi \in S^q_{F(\cdot,\varphi(\cdot),\varphi'(\cdot))}$.

Here $L^p(T)$ is a uniformly convex Banach space with the norm $\|x\|_p = (\int_0^b \|x(t)\|^p \mathrm{d}t)^{1/p}$, and $W^{1,p}(T)$ denotes a Sobolev space, that is $W^{1,p}(T) = \{x : T \to \mathbb{R} : x \text{ is absolutely continuous and } x' \in L^p(T)\}$ endowed with the norm $\|x\|_{1,p} = (\|x\|_p^p + \|x'\|_p^p)^{1/p}$. In addition, $C(T)$ (or $C^1(T)$) signifies a Banach space containing all continuous functions (or continuously differential functions) with the norm $\|x\|_0 = \max_{t\in T} |x(t)|$ (or $\|x\|_{C^1} = \|x\|_0 + \|x'\|_0$).

Secondly, we give briefly some notions and results from multivalued analysis and the theory of nonlinear operators of monotone type. All of them can be found in many books, such as Barbu [12], Hu and Papageorgiou [13], etc.

Let X, Y be Banach spaces, and let

$\mathscr{P}_0(X)$ be the family of all nonempty subsets of X, and similarly

$\mathscr{P}_{f(c)}(X) = \{A \subseteq X : A \text{ is nonempty and closed (convex)}\}$,

$\mathscr{P}_k(X) = \{A \subseteq X : A \text{ is nonempty and compact}\}$,

$\mathscr{P}_{fc}(X) = \{A \subseteq X : A \text{ is nonempty, closed and convex}\}$,

$\mathscr{P}_{kc}(X) = \{A \subseteq X : A \text{ is nonempty, compact and convex}\}$.

For a multivalued map $F : X \to \mathscr{P}_0(Y)$, we say it is upper semicontinuous (usc in short) if the inverse image $F^{-1}(C) = \{x \in X : F(x) \cap C \neq \emptyset\}$ is closed for any closed subset C of Y. In this case, if F has closed values, then its graph GrF is closed in $X \times Y$.

A operator $f : X \to Y$ is said to be completely continuous if for any sequence x_n converging to an element, say x, weakly in X, the corresponding sequence $f(x_n)$ converges to $f(x)$ strongly in Y. As we know that, if X is reflexive, then f is completely continuous implying that f is continuous and compact.

The following theorem, which is used as a theoretical basis in this paper, is a multivalued generalization of the Leray–Schauder alternative theorem, due to Bader [14], that is:

Theorem 1.1. *Let X, Y be Banach spaces, and $F : X \to P_{wkc}(Y)$ be usc from X into Y endowed with weak topology. Let $\psi : Y \to X$ be completely continuous. If $\Psi = \psi \circ F : X \to 2^X \setminus \{\emptyset\}$ is compact, then either*

(i) *the set $S = \{x \in X : x \in \mu\psi \circ Fx, 0 < \mu < 1\}$ is unbounded, or*

(ii) *Ψ has a fixed point, i.e. there exists a $x \in X$, s.t. $x \in \Psi x$.*