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Global existence of diffusive–dispersive traveling waves for general flux functions

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ABSTRACT

We establish a global existence of traveling waves for diffusive–dispersive conservation laws for locally Lipschitz flux functions. Using Lyapunov stability techniques, we reduce the global problem of finding traveling waves to considering local behaviors of a stable trajectory of the saddle point.

1. Introduction

We consider in this paper the existence of a certain kind of smooth solutions, called the *traveling waves*, of the following third-order partial differential equation

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = a\partial_{xx}u(x,t) + b\partial_{xxx}u(x,t), \quad x \in \mathbb{R}, t > 0, \tag{1.1}$$

where, a, b represent the diffusion and dispersion coefficients, respectively. Here, we assume that a and b are positive constants.

When traveling waves of (1.1) exist, one is interested in their limit when $a, b \to 0+$. This is a certain kind of admissibility criteria for shock waves of the conservation law

$$\partial_t u + \partial_x f(u) = 0. \tag{1.2}$$

Conversely, when a shock wave of (1.2) exists, it has been shown that the corresponding traveling waves also exist, under certain circumstances, see [1].

Diffusive–dispersive traveling waves have been studied by many authors, see [2–8], etc. In [1], the relationship between the existence of traveling waves of (1.1) and the existence of *classical* and *nonclassical* shock waves was considered. A geometrical distinction between the classical shocks and nonclassical shocks is that in the case of classical shocks, the line connecting the two left-hand and right-hand states does not cross the graph of the flux function in the interval between these two states, while it is the case for nonclassical shocks. The reader is referred to [9–15] for classical shocks, to [4,16,17,16,1,18–20] for nonclassical shock waves. Recently, non-monotone traveling waves for van der Waals fluids with diffusion and dispersion terms were obtained in [21].

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The present paper devotes to establishing a global existence of traveling waves of (1.1), where the flux function f is solely locally Lipschitz. Our strategy is as follows. First, we transform the problem of finding a traveling wave connecting a left-hand state u_- to a right-hand state u_+ to a 2×2 system of ordinary differential equations. Second, we consider the asymptotical behavior of trajectories of the two equilibria $(u_\pm, 0)$ of the system, which turn out to be a stable node and a saddle point. Third, we define a Lyapunov function in such a way that this function enables us to estimate the domain of attraction of the stable node. We then show that the saddle point is in fact on the boundary of the attraction domain of the node. Since a saddle point always admits stable trajectories, this raises the hope that a stable trajectory from the saddle would eventually enter the domain of attraction of the node. Whenever this happens, a connection between the stable node and the saddle is established. This also gives us a traveling wave connecting the states $u_\pm$. Finally, a sharp estimation of the domain of attraction of the node using Lyapunov function yields the existence result.

The organization of the paper is as follows. In Section 2, we will provide basic concepts and properties of traveling waves of (1.1) and the stability of equilibria of the associated differential equation. Furthermore, we will establish an invariance result concerning traveling waves of (1.1), relying on LaSalle's invariance principle. In Section 3 we will demonstrate that traveling waves of (1.1) exist whenever there is a Lax shock of the associate conservation law (1.2) satisfying Oleinik's entropy condition.

2. Traveling waves and stability of equilibria

Let us consider *traveling waves* of (1.1) i.e., smooth solution $u = u(y)$ depending on the re-scaled variable

$$y := \alpha \frac{x - \lambda t}{a} = \frac{x - \lambda t}{\sqrt{b}} \tag{2.1}$$

for some constant speed λ and

$$\alpha = a/\sqrt{b}.$$

Substituting $u = u(y)$ to (1.1), after re-scaling, the traveling wave u connecting a left-hand state u_- to a right-hand state u_+ satisfies the ordinary differential equation

$$-\lambda \frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}f(u)}{\mathrm{d}y} = \alpha \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} + \frac{\mathrm{d}^3 u}{\mathrm{d}y^3}, \quad y \in \mathbb{R}, \tag{2.2}$$

and the boundary conditions

$$\begin{aligned} &\lim_{y \to \pm\infty} u(y) = u_\pm, \\ &\lim_{y \to \pm\infty} \frac{\mathrm{d}u}{\mathrm{d}y} = \lim_{y \to \pm\infty} \frac{\mathrm{d}^2 u}{\mathrm{d}y^2} = 0. \end{aligned} \tag{2.3}$$

Integrating (2.2) and using the boundary condition (2.3), we find u such that

$$\frac{\mathrm{d}^2 u}{\mathrm{d}y^2} + \alpha \frac{\mathrm{d}u}{\mathrm{d}y} = -\lambda(u(y) - u_-) + f(u) - f(u_-), \quad y \in \mathbb{R}. \tag{2.4}$$

Using (2.3) again, we deduce from (2.4)

$$\lambda = \frac{f(u_+) - f(u_-)}{u_+ - u_-}. \tag{2.5}$$

Setting

$$v = \frac{\mathrm{d}u}{\mathrm{d}y}$$

we can re-write the second-order differential equation (2.4) to the following second-order system

$$\begin{aligned} \frac{\mathrm{d}u(y)}{\mathrm{d}y} &= v(y), \\ \frac{\mathrm{d}v(y)}{\mathrm{d}y} &= -\alpha v(y) - \lambda(u(y) - u_-) + f(u(y)) - f(u_-). \end{aligned} \tag{2.6}$$

The system (2.6) can be written in a more compact of autonomous differential equations

$$\frac{\mathrm{d}U(y)}{\mathrm{d}y} = F(U(y)), \quad y \in \mathbb{R}, \tag{2.7}$$