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On a p-Kirchhoff equation via Fountain Theorem and Dual Fountain Theorem ☆

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ABSTRACT

In this paper, we show the existence of infinite solutions to the Kirchhoff type quasilinear elliptic equation

$$\left[M\left(\int_\Omega (|\nabla u|^p + \lambda(x)\,|\,u\,|^p)\mathrm{d}x\right)\right]^{p-1}(-\Delta_p u + \lambda(x)\,|\,u\,|^{p-2}\,u) = f(x,u)$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ with nonlinear boundary condition $|\,\nabla u\,|^{p-2}\frac{\partial u}{\partial \nu} = \eta\,|\,u\,|^{p-2}$ on $\partial\Omega$. The method we used here is based on "Fountain Theorem" and "Dual Fountain Theorem".

1. Introduction

In this paper we deal with the nonlocal elliptic problem (P) of the p-Kirchhoff type given by

$$\left[M\left(\int_\Omega (|\nabla u|^p + \lambda(x)|u|^p)\mathrm{d}x\right)\right]^{p-1}(-\Delta_p u + \lambda(x)|u|^{p-2}u) = f(x,u), \quad x \in \Omega, \tag{1}$$

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \eta|u|^{p-2}, \quad x \in \partial\Omega. \tag{2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\frac{\partial}{\partial \nu}$ is the outer unit normal derivative, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < N$.

$$\lambda(x) \in L^\infty(\Omega) \quad \text{satisfying } \mathrm{essinf}_{x\in\bar{\Omega}}\lambda(x) > 0. \tag{3}$$

The function $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing function and there is a constant $m_0 > 0$ such that $M(t) \geq m_0$, for all $t \geq 0$.

The perturbation $f(x,t)$ is a Caratheodory function with subcritical growth with respect to t, that is, $|f(x,t)| \leq C(1 + |t|^{q-1})$ holds true for all $x \in \Omega$ and $t \in \mathbb{R}$ with $1 \leq q < p^* = \frac{Np}{N-p}$, where p^* is a critical exponent according to the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, η is a real parameter.

Problem (P) is called nonlocal because of the presence of the term M, which implies that the equation in (P) is no longer pointwise identities. This provokes some mathematical difficulties which makes the study of such a problem particulary

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interesting. This problem has a physical motivation. Similar operator $M(\int_\Omega |\nabla u|^2 \mathrm{d}x)\Delta u$ appears in the Kirchhoff equation, which arises in nonlinear vibrations, namely

$$\begin{cases} u_{tt} - M(\|u\|)\Delta u = f(x,u), & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & u_t(x,0) = u_1(x). \end{cases}$$

p-Kirchhoff problem began to attract the attention of several researchers mainly after the work of Lions [1] where a functional analysis approach was proposed to attack it.

The reader may consult [2–4,6–11] and the references therein, for similar problem in several cases.

Throughout this paper, by weak solutions of (P) we understand critical points of the associated energy functional φ acting on the Sobolev space $W^{1,p}(\Omega)$:

$$\varphi(u) = \frac{1}{p}\widehat{M}\left(\int_\Omega (|\nabla u|^p + \lambda(x)|u|^p)\mathrm{d}x\right) - \int_\Omega F(x,u)\mathrm{d}x - \frac{\eta}{p}\int_{\partial\Omega} |u|^p \mathrm{d}S, \tag{4}$$

where $\widehat{M}(t) = \int_0^t [M(s)]^{p-1}\mathrm{d}s$, $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$, and $\mathrm{d}S$ is the surface measure on the boundary. Obviously $\varphi(u) \in C^1(W^{1,p}(\Omega),\mathbb{R})$ and

$$\begin{aligned}\langle \varphi'(u), v\rangle = {} & \left[M\left(\int_\Omega (|\nabla u|^p + \lambda(x)|u|^p)\mathrm{d}x\right)\right]^{p-1} \int_\Omega [|\nabla u|^{p-2}\nabla u \nabla v \\ & + \lambda(x)|u|^{p-2}uv]\mathrm{d}x - \int_\Omega f(x,u)v - \eta\int_{\partial\Omega} |u|^{p-2}uv\mathrm{d}S\end{aligned}$$

for all $u, v \in W^{1,p}(\Omega)$, where

$$W^{1,p}(\Omega) = \left\{u \in L^p(\Omega) : \int_\Omega |\nabla u|^p \mathrm{d}x < \infty\right\}$$

is a Banach space with the norm

$$\|u\|_{1,p} := \left(\int_\Omega [|\nabla u|^p + \lambda(x)|u|^p]\mathrm{d}x\right)^{\frac{1}{p}} \quad \text{for } u \in W^{1,p}(\Omega).$$

Our main results are as following.

Theorem 1.1. *Suppose that $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous nondecreasing function satisfies the following conditions*

(m0) *there exists a constant $m_0 > 0$ such that $M(t) \geq m_0$ for all $t \geq 0$;*
(m1) *there exists a constant $m_1 > 0$ such that $M(t) \leq m_1$ for all $t > 0$ and $p^* > p(\frac{m_1}{m_0})^{p-1}$.*

Caratheodory function f satisfies

(f1) *For some $p < q < p^*$, there exists a constant $C > 0$ such that*

$$|f(x,t)| \leq C(1 + |t|^{q-1}) \quad \text{for all } x \in \Omega,\ t \in \mathbb{R};$$

(f2) *There exist $p^* > \alpha > p(\frac{m_1}{m_0})^{p-1}$ and $R > 0$ such that*

$$|t| \geq R \Rightarrow 0 < \alpha F(x,t) \leq tf(x,t) \quad \text{for all } x \in \Omega;$$

(f3) *$f(x,t)$ is odd with respect to t, that is,*

$$f(x,-t) = -f(x,t) \quad \text{for all } x \in \Omega,\ t \in \mathbb{R}.$$

Then there exists a constant $\Lambda > 0$ such that for any $\eta < \Lambda$, the problem (P) has a sequence of solutions $u_k \in W^{1,p}(\Omega)$ such that $\varphi(u_k) \to \infty$, as $k \to \infty$.

For a special f, we obtain a sequence of weak solutions with negative energy by dual fountain theorem.

Theorem 1.2. *M satisfies* (m0) (m1) *in Theorem 1.1, let $f(x,t) = \mu|t|^{r-2}t + \lambda|t|^{s-2}t$, where $1 < r < p < s < p^*$,. Then there exists a constant $\Lambda > 0$ such that for any $\eta < \Lambda$,*

(1) *For every $\lambda > 0$, $\mu \in \mathbb{R}$, problem (P) has a sequence of solutions u_k such that $\varphi(u_k) \to \infty$ as $k \to \infty$;*
(2) *For every $\mu > 0$, $\lambda \in \mathbb{R}$, problem (P) has a sequence of solutions v_k such that $\varphi(v_k) < 0$, $\varphi(v_k) \to 0$ as $k \to \infty$.*