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Chaotic sets of continuous and discontinuous maps

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ABSTRACT

This paper discusses Li–Yorke chaotic sets of continuous and discontinuous maps with particular emphasis to shift and subshift maps. Scrambled sets and maximal scrambled sets are introduced to characterize Li–Yorke chaotic sets. The orbit invariant for a scrambled set is discussed. Some properties about maximality, equivalence and uniqueness of maximal scrambled sets are also discussed. It is shown that for shift maps the set of all scrambled pairs has full measure and chaotic sets of some discontinuous maps, such as the Gauss map, interval exchange transformations, and a class of planar piecewise isometries, are studied. Finally, some open problems on scrambled sets are listed and remarked.

1. Introduction

The Li–Yorke chaos is still an interesting topic in dynamical systems, and it is still a challenging problem to characterize a Li–Yorke chaotic set of a continuous or even discontinuous map. The idea of a scrambled set was introduced by Smítal [1] to characterize Li–Yorke chaotic sets in a certain way. There have been extensive studies on scrambled sets of an iterated continuous map on a compact interval. There have been also some results on scrambled sets of continuous self-maps on higher dimensional cube I^n, $n \geq 2$ or more general spaces [2–5]. This paper discusses scrambled sets of continuous and some discontinuous maps with particular emphasis to shift maps.

Let $f : X \to X$ be a map (continuous or discontinuous) of a compact metric space X with metric d.

Definition 1.1. For two points $a, b \in X$, (a, b) is a scrambled pair for the map f if

$$\limsup_{n\to+\infty} d(f^n(a), f^n(b)) > 0, \tag{1.1}$$

$$\liminf_{n\to+\infty} d(f^n(a), f^n(b)) = 0. \tag{1.2}$$

A subset $S \subseteq X$ containing at least two points is a scrambled set [6] of f, if for any $a, b \in S$, $a \neq b$, (a, b) is a scrambled pair for f.

If a scrambled set S of f is also uncountable, we call S a chaotic set for f, and we say that f is chaotic (in the sense of Li–Yorke).

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For a continuous self-map f on a compact interval, the existence of a scrambled pair is sufficient for f to be chaotic in the sense of Li–Yorke [7]. For subshifts of finite type the same result is valid [8].

Remark 1.1. We recall that a pair (a, b) is called distal if $\liminf_{n\to+\infty} d(f^n(a), f^n(b)) > 0$; proximal if $\liminf_{n\to+\infty} d(f^n(a), f^n(b)) = 0$; asymptotic if $\lim_{n\to+\infty} d(f^n(a), f^n(b)) = 0$. So a non-distal pair is proximal, and a proximal pair is either asymptotic or scrambled.

Remark 1.2. The original characterization of chaos in the famous Li–Yorke theorem is via three conditions. The third one is:

$$\limsup_{n\to+\infty} d(f^n(x), f^n(p)) > 0,$$

for $\forall x \in S$ and for any periodic point $p \in P(f)$. But it is known that this condition is not essential and thus is removable [9].

Remark 1.3. The concepts scrambled pair and scrambled set are used to better understand the Li–Yorke type chaos. In [10] scrambled pair (a, b) is further required to be nonwandering. With this additional assumption on nonwandering, the Li–Yorke chaos for one-dimensional maps on interval is equivalent to topological chaos (we call a map f is topologically chaotic if f has positive topological entropy) [8]. The same conclusion holds for subshifts of finite type. Note that without this additional assumption, some maps may be chaotic in the sense of Li–Yorke but have zero topological entropy [11].

Definition 1.2. For an integer $p > 0$, $x \in X$ is p-scrambled, if for the orbit of x under f: $\mathrm{Orb}_f(x) = \{x_0 = x, x_1 = f(x), \ldots, x_n = f^n(x), \ldots\}$,

$$\limsup_{n\to+\infty} d(x_n, x_{n+p}) > 0, \tag{1.3}$$

$$\liminf_{n\to+\infty} d(x_n, x_{n+p}) = 0. \tag{1.4}$$

In Section 2 we will discuss p-scrambled points briefly.

Definition 1.3. We call α an orbit invariant on $S \subseteq X$ for f, if

(i) $\alpha : \bigcup_{n=0}^{+\infty} f^n(S) \to (0, 1)$ is a function;
(ii) $\alpha|_S$ is injective;
(iii) $\alpha(f^n(x)) = \alpha(x)$, $\forall x \in S$, $\forall n \geq 0$,

i.e., α has the same value on an orbit, and different values on different orbits.

We will show in Theorem 2.2 in Section 2 that under some conditions every scrambled set contains an orbit invariant.

Let $A = \{0, 1, \ldots, N-1\}$ for some integer $N \geq 2$ with the discrete metric d, denote by $\Sigma(N)$ the space consisting of one-sided sequences in A. So $x \in \Sigma(N)$ may be denoted by $x = (x_0, x_1, \ldots, x_i, \ldots)$, $x_i \in A$, $i \geq 0$. Let $\Sigma(N)$ be endowed with the product topology. Then $\Sigma(N)$ is metrizable, and the metric on $\Sigma(N)$ can be chosen to be

$$\rho(x, y) = \sum_{i=0}^{+\infty} \frac{d(x_i, y_i)}{2^i}, \quad x = (x_0, x_1, \ldots), y = (y_0, y_1, \ldots) \in \Sigma(N).$$

The shift map $\sigma\colon \Sigma(N) \to \Sigma(N)$ is defined by $(\sigma(x))_i = x_{i+1}$, $i = 0, 1, \ldots$. A two-sided shift map can be defined similarly, acting by the same formula but on doubly infinite sequences of elements of A. The shift map $\sigma\colon \Sigma(\mathbb{N}) \to \Sigma(\mathbb{N})$ over a countable alphabet $\mathbb{N}$ can be defined similarly by letting $A = \mathbb{N} = \{0, 1, \ldots, k, \ldots\}$.

Given f, what are the sufficient conditions for $S \subseteq X$ to be a scrambled (chaotic) set of f? When restrict our discussions to shift maps $\sigma : \Sigma(N) \to \Sigma(N)$, then we will show in Theorem 2.4 that the existence of an orbit invariant on S is also the main part of the sufficient conditions for S to be a scrambled set.

In Theorem 2.6 we will reveal some common properties of scrambled (chaotic) sets.

In Section 3 we discuss some problems related to maximal scrambled sets of shift maps, and show that there is no unique maximal scrambled set for full shifts up to the equivalence under Definition 3.2.

And as an extension of the topic of this paper, scrambled sets of some discontinuous maps, such as the Gauss map, interval exchange transformations, and a class of planar piecewise isometries, are studied in Section 4. As far as we know this is the first time the topic is studied.

Finally, in Section 5, we give some remarks and open problems on scrambled sets of shift maps and related topics.

2. Scrambled sets and orbit invariants

We first characterize scrambled sets by p-scrambled sets. Let $\mathrm{Orb}_f(x)$ be the orbit of f from x, i.e., $\mathrm{Orb}_f(x) = \{f^n(x) : n \geq 0\}$.

Proposition 2.1. *If for all $p > 0$, x is p-scrambled, then the orbit of f from x $\mathrm{Orb}_f(x)$ is a scrambled set.*