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Regular variation on measure chains

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ABSTRACT

In this paper we show how the recently introduced concept of regular variation on time scales (or measure chains) is related to a Karamata type definition. We also present characterization theorems and an embedding theorem for regularly varying functions defined on suitable subsets of reals. We demonstrate that for a "reasonable" theory of regular variation on time scales, certain additional condition on a graininess is needed, which cannot be omitted. We establish a number of elementary properties of regularly varying functions. As an application, we study the asymptotic properties of solution to second order dynamic equations.

1. Introduction

Recall that a measurable function $f : [a, \infty) \to (0, \infty)$ is said to be *regularly varying of index* ϑ, $\vartheta \in \mathbb{R}$, if it satisfies

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^{\vartheta} \quad \text{for all } \lambda > 0; \tag{1}$$

we write $f \in \mathcal{RV}_{\mathbb{R}}(\vartheta)$. If $\vartheta = 0$, then f is said to be *slowly varying*. Fundamental properties of regularly varying functions are that relation (1) holds uniformly on each compact λ-set in $(0, \infty)$ and $f \in \mathcal{RV}_{\mathbb{R}}(\vartheta)$ if and only if it may be written in the form $f(x) = \varphi(x)x^{\vartheta} \exp\left\{\int_a^x \eta(s)/s \, ds\right\}$, where φ and η are measurable with $\varphi(x) \to C \in (0, \infty)$ and $\eta(x) \to 0$ as $x \to \infty$; see [1–4]. In the basic theory of regularly varying sequences two main approaches are known. First, the approach by Karamata [5], based on a counterpart of the continuous definition: A positive sequence $\{f_k\}$, $k \in \{a, a+1, \dots\} \subset \mathbb{Z}$, is said to be *regularly varying of index* ϑ, $\vartheta \in \mathbb{R}$, if

$$\lim_{k\to\infty} \frac{f_{[\lambda k]}}{f_k} = \lambda^{\vartheta} \quad \text{for all } \lambda > 0, \tag{2}$$

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where $[u]$ denotes the integer part of u. Second, the approach by Galambos and Seneta [6], based on a purely sequential definition: A positive sequence $\{f_k\}$ is said to be *regularly varying of index* ϑ if there exists a positive sequence $\{\alpha_k\}$ satisfying $f_k \sim C\alpha_k$ and $\lim_{k\to\infty} k\,(1-\alpha_{k-1}/\alpha_k) = \vartheta$, C being a positive constant. In [7] it was shown that these two definitions are equivalent. In [8] we suggest to replace (equivalently) the second condition in the latter definition by $\lim_{k\to\infty} k\Delta\alpha_k/\alpha_k = \vartheta$. A regularly varying sequence can be represented as $f_k = \varphi_k k^{\vartheta} \prod_{j=a}^{k-1}\left(1+\psi_j/j\right)$, see [8], or as $f_k = \varphi_k k^{\vartheta} \exp\left\{\sum_{j=a}^{k-1} \psi_j/j\right\}$, where $\varphi_k \to C \in (0,\infty)$ and $\psi_k \to 0$ as $k\to\infty$, see [6,7]. For further reading on the discrete case we refer, e.g., to [9]. Recall that the theory of regular variation can be viewed as the study of relations similar to (1) or (2), together with their wide applications, see, e.g., [1,2,4,8,10–12]. There is a very practical way how regularly varying functions can be understood: Extension in a logical and useful manner of the class of functions whose asymptotic behavior is that of a power function, to functions where asymptotic behavior is that of a power function multiplied by a factor which varies "more slowly" than a power function. In [6,7], see also [13], the so-called embedding theorem was established (and the converse result holds as well): If $\{y_k\}$ is a regularly varying sequence, then the function R (of a real variable), defined by $R(x) = y_{[x]}$, is regularly varying. Such a result makes it then possible to apply the continuous theory to the theory of regularly varying sequences. However, the development of a discrete theory, analogous to the continuous one, is not generally close, and sometimes far from a simple imitation of arguments for regularly varying functions, as noticed and demonstrated in [7]. Simply, the embedding theorem is just one of powerful tools, but sometimes it is not immediate that from a continuous results its discrete counterpart is easily obtained thanks to the embedding; sometimes it is even not possible to use this tool and the discrete theory requires a specific approach, different from the continuous one.

Recall that the calculus on time scales (or, more generally, on measure chains) deals essentially with functions defined on nonempty closed subsets of $\mathbb{R}$, see [14,15]. Hence, it unifies and extends usual calculus and quantum (q- or h-) calculi. Also it helps to describe and understand discrepancies between individual cases.

A theory of regular variation on time scales offers something more than the embedding result, and has the following advantages: Once there is proved a result on a general time scale, it automatically holds for the continuous and the discrete case, without any other effort. Moreover, at the same time, the theory works also on other time scales which may be different from the "classical" ones.

In [16] we introduced the concept of regular variation on time scales, the form of which is motivated by a modification of the purely sequential criterion mentioned above. There we also derived a simple characterization of regularly varying functions, and investigated regularly varying behavior of decreasing solutions to a second order linear dynamic equation. In the present paper we show how the definition from [16] is connected to a Karamata type definition, where the latter one is motivated by (1) and (2). Further, under certain conditions, we establish an embedding result which relates a general time scales theory with the continuous theory. We also derive two important representation formulas and state a number of useful elementary properties of regularly varying functions on time scales. We show that to obtain a reasonable theory, from a certain point of view, we need an additional requirement on the graininess of a time scale, which cannot be improved. Roughly speaking, if the graininess is sufficiently small ($\mu(t) = o(t)$), then we get a continuous like (or a discrete like) theory. If the graininess is as in the q-calculus, then, as shown in [17], not only that basic results are differently looking, but in some important aspects we got a surprising simplifications comparing with the "classical" theories. Finally, for grainnesses which are somehow large, the theory fails.

As an application, we study the asymptotic properties of solutions of second order dynamic equations.

2. Preliminaries

We assume that the reader is familiar with the notion of time scales. Thus note just that $\mathbb{T}, \sigma, f^{\sigma}, \mu, f^{\Delta}, \int_a^b f^{\Delta}(s)\,\Delta s$, and $e_f(t,a)$ stand for time scale, forward jump operator, $f\circ\sigma$, graininess, delta derivative of f, delta integral of f from a to b, and generalized exponential function, respectively. Recall that the concept of integration on time scales can be developed in various usual manners, e.g., Newton, or Riemann, or Lebesgue. See [14], which is the initiating paper of the time scale theory, and the monograph [15] containing a lot of information on time scale calculus.

Before we give the first definition, note that in some parts below the conditions on smoothness can be somehow relaxed. But we do not do it since our theory focuses on a generalization in the sense of a "domain of definition" rather than considering "badly behaving" functions. In this paper, $\mathbb{T}$ is assumed to be unbounded above. In [16] we introduced the concept of regular variation on $\mathbb{T}$ in the following way.

Definition 1. A measurable function $f : \mathbb{T} \to (0,\infty)$ is said to be *regularly varying of index* ϑ, $\vartheta \in \mathbb{R}$, if there exists a positive *rd*-continuously delta differentiable function α satisfying

$$f(t) \sim C\alpha(t) \quad \text{and} \quad \lim_{t\to\infty} \frac{t\alpha^{\Delta}(t)}{\alpha(t)} = \vartheta, \tag{3}$$

C being a positive constant; we write $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$. If $\vartheta = 0$, then f is said to be *slowly varying*, we write $f \in \mathcal{SV}_{\mathbb{T}}$.

Using elementary properties of linear first order dynamic equations and generalized exponential functions, the following representation was established in [16]: $f \in \mathcal{RV}_{\mathbb{T}}(\vartheta)$ if and only if it has the representation

$$f(t) = \varphi(t)e_{\delta}(t,a), \tag{4}$$