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A general iterative algorithm for nonexpansive mappings in Hilbert spaces

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ABSTRACT

Let H be a real Hilbert space. Suppose that T is a nonexpansive mapping on H with a fixed point, f is a contraction on H with coefficient $0 < \alpha < 1$, and $F : H \to H$ is a k-Lipschitzian and η-strongly monotone operator with $k > 0, \eta > 0$. Let $0 < \mu < 2\eta/k^2, 0 < \gamma < \mu(\eta - \frac{\mu k^2}{2})/\alpha = \tau/\alpha$. We proved that the sequence $\{x_n\}$ generated by the iterative method $x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n$ converges strongly to a fixed point $\tilde{x} \in F_{ix}(T)$, which solves the variational inequality $\langle(\gamma f - \mu F)\tilde{x}, x - \tilde{x}\rangle \le 0$, for $x \in F_{ix}(T)$.

1. Introduction

Let H be a real Hilbert space with inner product $\langle , \rangle$ and induced norm $\| \cdot \|$. $T : H \to H$ is nonexpansive if $\|TX - Ty\| \le \|x - y\|$ for all $x, y \in H$. The set of fixed points of T is the set $F_{ix}(T) := \{x \in H : Tx = x\}$. We assume that $F_{ix}(T) \neq \phi$, it is well known that $F_{ix}(T)$ is closed convex.

Moudafi [1] introduced the viscosity approximation method for nonexpansive mappings. Let f be a contraction on H, starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \ge 0, \tag{1}$$

where $\{\alpha_n\}$ is a sequence in (0, 1). Xu [2] proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle(I - f)x^*, x - x^*\rangle \ge 0, \quad \text{for } x \in F_{ix}(T), \tag{2}$$

where $C = F_{ix}(T)$. We all know that iterative methods for nonexpansive mappings can be used to solve a convex minimization problem. See, e.g., [3–5] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{3}$$

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where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H. Assume A is strongly positive bounded linear operator. That is, there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \text{for all } x \in H. \tag{4}$$

In [4], it is proved that the sequence $\{x_n\}$ defined by the iterative method below with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n, \quad n \geq 0, \tag{5}$$

converges strongly to the unique solution of the minimization problem (3) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Combining the iterative method (1) and (5), Marino and Xu [6] consider the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0, \tag{6}$$

it is proved that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (6) converges strongly to the unique solution of the variational inequality

$$\langle(\gamma f - A)\tilde{x}, x - \tilde{x}\rangle \leq 0 \quad x \in C, \tag{7}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x = \gamma f(x)$ for $x \in H)$). Some people also study the applications of the iterative method (6) [7,8].

On the other hand, Yamada [5] introduced the following hybrid iterative method for solving the variational inequality

$$x_{n+1} = Tx_n - \mu\lambda_n F(Tx_n), \quad n \geq 0, \tag{8}$$

where F is k-Lipschitzian and η-strongly monotone operator with $k > 0, \eta > 0, 0 < \mu < 2\eta/k^2$, then he proved that if $\{\lambda_n\}$ satisfying appropriate conditions, the $\{x_n\}$ generated by (8) converges strongly to the unique solution of variational inequality

$$\langle F\tilde{x}, x - \tilde{x}\rangle \geq 0, \quad x \in F_{ix}(T).$$

In this paper, we will combine the iterative method (6) with the Yamada's method (8) and consider the following general iterative method

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n, \quad n \geq 0, \tag{9}$$

we will prove that if the sequence $\{\alpha_n\}$ of parameters satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (9) converges strongly to the unique solution $x^* \in C$ of the variational inequality

$$\langle(\gamma f - \mu F)\tilde{x}, x - \tilde{x}\rangle \leq 0, \quad \text{for } x \in C,$$

where $C = F_{ix}(T)$, our results improve and extend the corresponding results given by Marino, Xu and Yamada.

2. Preliminaries

Throughout this paper, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. The following lemmas are useful for our paper.

Lemma 2.1 ([9]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([10]). *Let H be a Hilbert space, K a closed convex subset of H, and $T : K \to K$ a nonexpansive mapping with $F_{ix}(T) \neq \phi$, if $\{x_n\}$ is a sequence in K weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

The following lemmas are easy to prove.