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Uniform boundedness in time and large time behavior of weak solutions to a kind of dissipative system

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This paper deals with the uniform boundedness (as well as the existence) and large time behavior of the weak entropy solutions to a kind of compressible Euler equation with dissipation effect. The existence and uniform boundedness in time of weak solutions are proved by using the Lax–Friedrichs scheme and compensate compactness. Time asymptotically, the density is showed to satisfy a kind of nonlinear Fokker–Planck equation and the momentum obeys to the Darcy's law. As a by product, the exponentially decay rate is obtained.

1. Introduction

We consider the compressible Euler equation with dissipation effect:

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x \left(\dfrac{m^2}{\rho} + P(\rho) \right) = \alpha\rho - \beta m. \end{cases} \tag{1}$$

This system is a model of the fluid movement that takes place in a vertical column of the medium. Here ρ, m, and P denote the density, momentum and pressure; the constants $\alpha > 0, \beta > 0$ represent gravity and friction. Assuming the flow is a polytropic perfect gas, then $P(\rho) = P_0\rho^\gamma,\ \gamma > 1$, with P_0 a positive constant, and γ the adiabatic gas exponent. Without loss of generality, we take $P_0 = \alpha = \beta = 1$ throughout this paper. System (1) is supplemented by the following initial value and boundary conditions:

$$\begin{cases} \rho(x, 0) = \rho_0(x) \geq 0, \qquad m(x, 0) = m_0(x), \quad 0 < x < 1, \\ m(0, t) = 0, \qquad m(1, t) = 0, \quad t \geq 0, \\ \displaystyle\int_0^1 \rho_0(x)\mathrm{d}x = \rho^* > 0, \end{cases} \tag{2}$$

where $m_0(x)$ satisfy the following compatibility condition $m_0(0) = m_0(1) = 0$, and the last condition is to avoid the trivial case, $\rho \equiv 0$.

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The essential features of problems (1)–(2) are the nonstrictly hyperbolicity, that is, a pair of wave speed coalesce on the vacuum $\rho = 0$, and the dissipation mechanism. When the initial data is small smooth and is away from vacuum, problems (1)–(2) have global smooth solutions. However, when initial data is large or rough, the dissipation mechanism cannot hold the development of shock waves, and one has to consider weak entropy solutions. In this paper, we will first prove the existence and uniform boundedness in time of the L^∞ weak entropy solutions to (1)–(2), and then consider the large time behavior of the weak entropy solutions.

The existence of weak entropy solutions will be achieved by using the Lax–Friedrichs scheme and the compensate compactness frameworks established by [1–4]. Intensive literature has used this method to prove the existence of L^∞ weak entropy solutions; see [5–8]. It is worth pointing out that all these literature only get the weak entropy solutions that satisfy

$$\|\rho(x,t)\| \le C(T), \qquad \|m(x,t)\| \le C(T),$$

where $C(T)$ depends on time T. Here, one tip is given to obtain that the weak entropy solutions of problem (1)–(2) are uniformly bounded *in time*. For the large time behavior, it is conjectured that (1) is approximated by the decoupled system

$$\begin{cases} \bar\rho_t = P(\bar\rho)_{xx} - \bar\rho_x, \\ \bar m = \bar\rho - P(\bar\rho)_x. \end{cases} \tag{3}$$

The first equation is a kind of Fokker–Planck equation, while the second one states Darcy's law. The initial-boundary conditions turn into

$$\begin{cases} \bar\rho(x,0) = \bar\rho_0(x), \quad x \in [0,1], \\ (\bar\rho - P(\bar\rho)_x)(0,t) = (\bar\rho - P(\bar\rho)_x)(1,t) = 0, \quad t \ge 0, \end{cases} \tag{4}$$

correspondingly. To prove this conjecture, we take two steps: 1. adopt the framework (introduced by [9–11]) based on entropy dissipation to get the weak entropy solutions to problems (1), (2) that converge to a steady state exponentially fast in time; 2. use the energy estimate to prove problems (3), (4) that also converge to the same steady state exponentially. Hence, by the triangular inequality we obtain our aim. This idea is used in [12], too.

Before formulating the main results, we give the definition of weak entropy solutions to problems (1), (2).

Definition 1. For every $T > 0$, we define a *weak solution* of (1), (2) to be a pair of bounded measurable functions $v(x,t) = (\rho(x,t), m(x,t))$ satisfying the following pair of integral identities:

$$\begin{cases} \displaystyle\int_0^T\int_0^1 (\rho\varphi_t + m\varphi_x)\mathrm{d}x\mathrm{d}t + \int_0^1 \rho_0(x)\varphi(x,0)\mathrm{d}x = 0, \\ \displaystyle\int_0^T\int_0^1 \left(m\varphi_t + \left(\frac{m^2}{\rho} + P(\rho)\right)\varphi_x\right)\mathrm{d}x\mathrm{d}t + \int_0^T\int_0^1 (\rho - m)\varphi\mathrm{d}x\mathrm{d}t + \int_0^1 m_0(x)\varphi(x,0)\mathrm{d}x = 0, \end{cases} \tag{5}$$

for all $\varphi \in C_0^\infty(I_T)$ satisfying $\varphi(x,T) = 0$ for $0 \le x \le 1$ and $\varphi(0,t) = \varphi(1,t) = 0$ for $t \ge 0$, where $I_T = (0,1) \times [0,T)$, and m/ρ vanishes when $\rho = 0$. Moreover, m satisfies the boundary condition $(2)_2$ in the sense of trace, defined in [12].

Definition 2. The weak solution $v(x,t) = (\rho(x,t), m(x,t))$ is said to be a *weak entropy solution* of problems (1), (2) if it satisfies the entropy inequality

$$\eta_{et} + q_{ex} + \frac{m^2}{\rho} - m \le 0, \tag{6}$$

in the sense of distribution, where (η_e, q_e) are mechanical entropy–entropy–flux pair satisfying

$$\eta_e(\rho,m) = \frac{m^2}{2\rho} + \frac{\rho^\gamma}{\gamma-1}, \qquad q_e(\rho,m) = \frac{m^3}{2\rho^2} + \frac{\gamma}{\gamma-1}\rho^{\gamma-1}m. \tag{7}$$

Like the porous media equation, problem $(3)_1$ is a degenerate parabolic equation. At points (t,x) where $u > 0$, it is parabolic, but at points where $u = 0$, it is not. In fact, between a region where $u > 0$ and $u = 0$, u need not be smooth. Therefore, to problems (3), (4), it is necessary to consider the weak solution. Among the different notions of weak solutions, we follow the one introduced in [13,14]. Here we omit the details.

The main results of this paper are:

Theorem 1. *Suppose the initial data $\rho_0(x), m_0(x)$ satisfy*

$$0 \le \rho_0(x) \le C_0, \quad |m_0(x)| \le C_0\rho_0(x), \quad \text{a.e. } x \in [0,1]. \tag{8}$$

Then there exists a global weak entropy solution $(\rho(x,t), m(x,t))$ of problems (1), (2) in the region I_T, satisfying

$$0 \le \rho(x,t) \le C, \quad |m(x,t)| \le C\rho(x,t) \quad \text{a.e. } (x,t) \in I_T, \tag{9}$$

where C_0 and C are two constants.