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# Rigorous derivation of incompressible type Euler equations from non-isentropic Euler–Maxwell equations

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ABSTRACT

In this paper, we investigate the convergence of the time-dependent and non-isentropic Euler–Maxwell equations to incompressible Euler equations in a torus via the combined quasi-neutral and non-relativistic limit. For well prepared initial data, the convergences of solutions of the former to the solutions of the latter are justified rigorously by an analysis of asymptotic expansions and energy method.



## 1. Introduction

We investigate the non-relativistic limit problem for the (rescaled) non-isentropic Euler–Maxwell systems, which takes the following (non-conservative) form [1–4]:

$$\partial_t n + \operatorname{div}(nu) = 0, \tag{1.1}$$

$$\partial_t u + (u\cdot\nabla)u + \nabla\theta + \theta\nabla\ln n + u = -(E + \gamma u\times B), \tag{1.2}$$

$$\partial_t\theta + u\cdot\nabla\theta + \frac{2}{3}\theta\operatorname{div} u = \frac{1}{3}|u|^2 - (\theta-\theta_*), \tag{1.3}$$

$$\epsilon\gamma\partial_t E - \nabla\times B = \gamma nu, \qquad \gamma\partial_t B + \nabla\times E = 0, \tag{1.4}$$

$$\varepsilon\operatorname{div} E = b(x,t) - n, \qquad \operatorname{div} B = 0, \tag{1.5}$$

for $(x,t)\in\mathcal{T}^3\times[0,T]$.

Here, $n$, $u$, $\theta$ denote the scaled macroscopic density, mean velocity vector and temperature of the electrons and $E$, $B$ the scaled electric field and magnetic field. They are functions of a three-dimensional position vector $x\in\mathcal{T}^3$ and of the time $t>0$, where $\mathcal{T}^3 = (\frac{\mathbb{R}}{2\pi\mathbb{Z}})^3$ is the three-dimensional torus. The function $\theta_*(x)$ is the ambient device temperature and $b(x,t)$ stands for the prescribed density of positive charged background ions (doping profile). The fields $E$ and $B$ are coupled to

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the particles through the Maxwell equations and act on the particles via the Lorentz force $E + \gamma u \times B$. In the system (1.1)–(1.5), $j = nu$ stands for the current densities for the particles. Eqs. (1.1) and (1.2) are the mass and momentum balance laws respectively, while (1.4) and (1.5) are the Maxwell equations. It is well known that divergence equations (1.5) are redundant with Eqs. (1.4) as soon as they are satisfied by the initial data. However, we keep them in the system because this redundancy may be lost in the asymptotic limit.

Physically, the dimensionless parameter $\epsilon$ and $\gamma > 0$ are proportional to the Debye length and $\frac{1}{c}$ can be chosen independently on each other, according to the desired scaling (see [5]), where $c = (\epsilon_0 \nu_0)^{-\frac{1}{2}}$ is the speed of light, with $\epsilon_0$ and $\nu_0$ being the vacuum permittivity and permeability. It is small compared to the physical size of the known variables. Thus, the limit $\epsilon \to 0$ is called the quasi-neutral limit while the limit $\gamma \to 0$ is called the non-relativistic limit. In the present paper, we will concentrate on the so-called quasi-neutral regime. More precisely, we consider the combined quasi-neutral and non-relativistic limit of the Euler–Maxwell system in the following scaling case:

$$\gamma = \epsilon \to 0. \tag{1.6}$$

This scaling corresponds to a limit, without any magnetic field, coupled with a quasi-neutral limit.

There have been a lot of studies on the Euler–Poisson equations and their asymptotic analysis, in contrast to the case for study of the Euler–Maxwell equations. See [5–14] and the references therein. The first mathematical study of the Euler–Maxwell equations with extra relaxation terms is due to Chen et al. [15], where a global existence result for weak solutions in one-dimensional case is established by the fractional step Godunov scheme together with a compensated compactness argument. In 2003, J.W. Jerome established a local smooth solution theory for the Cauchy problem of compressible Hydrodynamic-Maxwell systems (Ref. [3]) via a modification of the classical semigroup-resolvent approach of Kato. Recently, the convergence of one-fluid (isentropic) Euler–Maxwell system to a compressible Euler–Poisson system was proven by Peng and Wang in [16] via the non-relativistic limit. Peng and Wang also proved that the combined non-relativistic and quasi-neutral limit is the (isentropic) incompressible Euler equation in a uniform background of non-moving ions with fixed unit density (see [17]). Furthermore, Peng and Wang derive incompressible e-MHD equations from compressible Euler–Maxwell equations via the quasi-neutral regime (see [18]). Yang and Wang [19,20] study the asymptotic limit of the two-fluid Euler–Maxwell system and non-isentropic Euler–Maxwell system via a non-relativistic regime.

The aim of this paper is to study the combined quasi-neutral and non-relativistic limit by the method of asymptotic expansions to the Cauchy problem for the multidimensional non-isentropic Euler–Maxwell models for plasmas or semiconductors. We formally derive an incompressible type of non-isentropic Euler system for electron velocity, entropy and the electrostatic potential. Noting that the ion density is constant, the limits of the electron velocity, entropy and the electrostatic potential satisfy the classical incompressible non-isentropic Euler equations. Under the assumption that the initial densities satisfy certain compatibility conditions which guarantee that no initial layer is formed, we obtain the existence of the asymptotic expansion and rigorously justify the formal limit for a periodic initial value by adapting the approach developed in [21–23]. The uniform error estimates are given with respect to $\gamma$ (or $\epsilon$) for each variable. It is noted that the initial value of the electron density cannot be given arbitrarily, and it should be determined by the initial data for the velocity, temperature and electric potential.

This remainder of this paper is organized as follows. In the next section, by means of formal asymptotic analysis we derive non-isentropic Euler equations of an incompressible type for the leading profiles of the expansion and corresponding linearized equations for the other profiles. In Section 3, the existence of the expansion is proved by solving an incompressible type Euler equation and linearized equations. Section 4 is devoted to rigorously justifying the asymptotic expansions developed in Section 2 up to any order under the condition that the initial expansions are well prepared, which excludes the formation of initial layers.

*Notation and preliminary results*

(1) Throughout this paper, $\nabla = \nabla_x$ is the gradient, $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index and $H^s(\mathcal{T}^3)$ denotes the standard Sobolev space in torus $\mathcal{T}^3$, which is defined by a Fourier transform, namely, $f \in H^s(\mathcal{T}^3)$ if and only if

$$\|f\|_s^2 = (2\pi)^d \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^s |(\mathcal{F}f)(k)|^2 < +\infty,$$

where $(\mathcal{F}f)(k) = \int_{\mathcal{T}^3} f(x) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x$ is the Fourier transform of $f \in H^s(\mathcal{T}^3)$. Note that if $\int_{\mathcal{T}^3} f(x)\mathrm{d}x = 0$, then $\|f\|_{L^2(\mathcal{T}^3)} \leq \|\nabla f\|_{L^2(\mathcal{T}^3)}$.

(2) Recall the following basic Moser-type calculus inequalities [24–26]: for $f, g, v \in H^s$ and any non-negative multi-index $\alpha$, $|\alpha| \leq s$,

$$\text{(i) } \|D_x^\alpha (fg)\|_{L^2} \leq C_s (\|f\|_{L^\infty} \|D_x^s g\|_{L^2} + \|g\|_{L^\infty} \|D_x^s f\|_{L^2}), \quad s \geq 0; \tag{1.7}$$

$$\text{(ii) } \|D_x^\alpha (fg) - f D_x^\alpha g\|_{L^2} \leq C_s (\|D_x f\|_{L^\infty} \|D_x^{s-1} g\|_{L^2} + \|g\|_{L^\infty} \|D_x^s f\|_{L^2}), \quad s \geq 1. \tag{1.8}$$

(3) (*Sobolev's inequality*.) For $s > \frac{d}{2}$, $\|f\|_{L^\infty} \leq C_s \|f\|_s$.

(4) If $s > \frac{d}{2}$, then for $f, g \in H^s$ and $|\alpha| \leq s$, $\|D_x^\alpha (fg)\|_{L^2} \leq C_s \|f\|_s \|g\|_s$.

(5) The same letter $C$ denotes various positive constants which do not depend on $t$, $\tau$ and the initial data.