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Asymptotic behavior of a class of nonlinear evolution equations[☆]

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ABSTRACT

In this paper, using a new method (or technology), we prove the existence and upper semi-continuity of the global attractors $\mathscr{A}_\omega$ for a class of nonlinear evolution equations in $D(A) \times D(A)$, where the nonlinear term f satisfies a critical exponential growth condition.

1. Introduction

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Given some $\omega > 0$, we study the following initial-boundary value problem for $u : \Omega \times \mathbf{R}^+ \to \mathbf{R}$:

$$\begin{cases} u_{tt} - \Delta u - \Delta u_t - \omega \Delta u_{tt} + f(u) = g & \text{in } \Omega \times \mathbf{R}^+, \\ u|_{t=0} = u_0, \ u_t|_{t=0} = u_1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \times \mathbf{R}^+, \end{cases} \tag{1.1}$$

where $g = g(x) \in L^2(\Omega)$ is given, $f \in C^2(\mathbf{R}, \mathbf{R})$ satisfies the following assumptions:

$$\liminf_{|s|\to\infty} f'(s) \geq -\lambda, \tag{1.2}$$

where λ is a positive constant and $\lambda < \lambda_1$ (λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ with the Dirichlet boundary condition). In particular, (1.2) implies that

$$f'(s) \geq -\beta, \quad \text{for any } s \in \mathbf{R}, \tag{1.3}$$

for some positive constant β.

$$|f''(s)| \leq C(1 + |s|^3), \quad \text{for any } s \in \mathbf{R}, \tag{1.4}$$

where C is positive constant.

We denote by F the function

$$F(u) = \int_0^u f(s)\mathrm{d}s.$$

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Eq. (1.1), which appears as a class of nonlinear evolution equations, like the strain solitary wave equation and dispersive-dissipative wave equation, is used to represent the propagation problems of a lengthwise-wave in nonlinear Elastic rods and Ion-sonic of space transformation by weak nonlinear effect, see for instance [1–4]. For the class of nonlinear evolution equations, the existence of global solutions has been studied in [5,6] etc. In [5], the author has discussed the existence of global strong solutions in $D(A) \times D(A)$. In [7], the author has obtained the existence of global attractors for Eq. (1.1) in $H_0^1(\Omega) \times H_0^1(\Omega)$. However, as we know, the long-time behaviors of strong solutions of problem (1.1) have not been considered up to now. In this paper, we try to discuss the problem.

The existence of global attractors for the classical wave equations in $H_0^1(\Omega) \times L^2(\Omega)$ (as $\omega = 0$) has been studied extensively in many monographs and lectures, for example, see [8–11] etc. In those references, authors usually change the classical wave equations into the following system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} + A\begin{pmatrix} u \\ v \end{pmatrix} + F\begin{pmatrix} u \\ v \end{pmatrix} = 0, \tag{1.5}$$

where

$$v = u_t, \quad A = \begin{pmatrix} 0 & -I \\ -\Delta & \delta I \end{pmatrix} \quad \textit{and} \quad F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ f(u) \end{pmatrix}.$$

Then the existence of global attractors follows from the classical semigroup theory. Since Eq. (1.1) contains the term Δu_{tt}, it is essentially different from the usual wave-type equations. Hence it cannot be directly reformulated as (1.5). So it is very difficult to study the existence of global attractors of Eq. (1.1) by the semigroup theory. It is also well known that the compact Sobolev embedding can be applied to obtain the existence of global attractor as the solution of the equation has higher regularity, e.g., although the initial data only belong to a weaker topology space, the solution will belong to a stronger topology space with higher regularity. However, since Eq. (1.1) contains the term Δu_{tt}, as the initial data u_0, u_1 belong to $D(A)$, the solution $u(t,x)$, $u_t(t,x)$ are always in $D(A)$.

It is well known that if we want to prove the existence of global attractors, the key point is to obtain the dissipation and the compactness of the semigroup in some sense. The key idea of the paper is to prove the dissipative feature of semigroup $\{S_\omega(t)\}_{t\geq 0}$ in $D(A) \times D(A)$ by using a new method (or technology) Lemma 2.6. By verifying the convenient condition, i.e. *Condition*(C) introduced in [12], we obtain the necessary compactness of the semigroup $\{S_\omega(t)\}_{t\geq 0}$ in $D(A) \times D(A)$. So we obtain global attractors of the product space. The other aim of the present work is to give positive answers to about regularity and upper semi-continuity for the attractors associated with strong solutions. Furthermore, we can delete the following assumption for nonlinearity of usually wave equations if using our method: there exists $c > 0$ such that

$$\liminf_{|s|\to\infty} \frac{cF(s) - sf(s)}{s^2} \geq 0.$$

In what follows, we give some notations which will be used throughout this paper. Let Ω be a bounded subset of $\mathbf{R}^3$ with a sufficiently smooth boundary, $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ with the corresponding norms $\|u\| = \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}$, $|u| = \left(\int_\Omega |u|^2\right)^{\frac{1}{2}}$, and $|\Delta u| = \left(\int_\Omega |\Delta u|^2\right)^{\frac{1}{2}}$ respectively. The norms in $L^p(\Omega)(3 \leq p < \infty)$ are denoted by $|u|_p = \left(\int_\Omega |u|^p\right)^{\frac{1}{p}}$, the scalar product of V, H are denoted by

$$((u, v)) = \int_\Omega \nabla u \nabla v, \qquad (u, v) = \int_\Omega uv,$$

respectively.

Then we introduce the product Hilbert spaces

$$E_0 = V \times V \quad \text{and} \quad E_1 = D(A) \times D(A).$$

Denote by C any positive constant which may be different from line to line and even in the same line.

2. Abstract results

In this subsection, we recall some basic concepts about the global attractors and recapitulate some abstract results about the existence of global attractors.

Definition 2.1 (*[13,14]*). Let X be a Banach space and $\{S(t)\}_{t\geq 0}$ be a family of operators on X. We say that $\{S(t)\}_{t\geq 0}$ is norm-to-weak continuous semigroup on X, if $\{S(t)\}_{t\geq 0}$ satisfies:

(i) $S(0) = Id$ (the identity);
(ii) $S(t)S(s) = S(t+s)\ \forall\, t, s \geq 0$;
(iii) $S(t_n)x_n \rightharpoonup S(t)x$ if $t_n \to t$ and $x_n \to x$ in X.

In evolution equation, this kind of semigroup corresponds to the solution that only satisfies weaker stability, and generally, it is neither continuous (i.e., norm-to-norm) nor weak continuous (i.e., weak-to-weak). But obviously, the