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Boundedness of orbits and stability of closed sets[☆]

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ABSTRACT

This paper explores fundamental connections between boundedness of orbits, and stability and attractivity of closed sets. For this purpose, the paper considers topological notions of stability and attractivity which do not depend on a metric. The notions considered are characterized in terms of restricted prolongations and positive limit sets, and connections with boundedness are studied. Finally, the notion of nontangency is used to give a Lyapunov result for Lyapunov stability, attractivity and boundedness of orbits. Unlike previous sufficient conditions for boundedness, our result does not require the Lyapunov function to be proper or weakly proper. Examples are provided to illustrate the results.

1. Introduction

Sufficient conditions used for Lyapunov analysis of stability and convergence often require verifying that trajectories are bounded. A classic example is LaSalle's invariant set theorem [1, p. 24], [2, Thm. 3.4], [3, Thm. 4.4], [4], which states that if a dynamical system admits a function V that is nonincreasing along the trajectories, then every bounded trajectory of the dynamical system converges to the largest invariant subset of the zero level set of the derivative $\dot{V}$. Recent sufficient conditions from [5] for convergence of trajectories of systems having a continuum of equilibria also involve first verifying boundedness of trajectories. Boundedness of trajectories is also of interest in the case of adaptive controllers involving feedback gains that evolve in response to plant behavior [6,7].

A classical sufficient condition for boundedness of trajectories involves functions that have compact sublevel sets, that is, functions that are radially unbounded or proper [8, Thm. 2], [2, p. 241], [4, Thm. 4], [9], [10, Thm. 8.7]. This condition states that if a proper Lyapunov function is nonincreasing along the trajectories, then every trajectory is bounded. More recently, [5] recognized that the same conclusion follows even if the Lyapunov function is assumed to be only weakly proper, that is, its (possibly noncompact) sublevel sets have compact connected components. In this paper, we present a sufficient condition for boundedness that does not require the Lyapunov function to be proper or weakly proper.

Boundedness of trajectories is fundamentally related to stability and attractivity. For instance, the lowest level set of a proper Lyapunov function that is bounded below and nonincreasing along the trajectories is compact and Lyapunov stable. Conversely, every compact Lyapunov stable set has a neighborhood of initial conditions with bounded orbits. Likewise, all

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bounded trajectories are attracted to the set formed by the union of all positive limit sets. Conversely, as our results will show, all trajectories that are attracted in a suitably strong sense to a closed invariant set have to be bounded. These observations suggest that one possible way of obtaining weaker sufficient conditions for boundedness is to relate boundedness of orbits to stability of level sets of Lyapunov functions. Since, in general, level sets of Lyapunov functions are noncompact, one has to consider stability of closed sets that may not necessarily be compact.

In much of the prior work on stability of closed, noncompact sets, stability and attractivity are defined in terms of a distance metric. See, for example, [11–13]. Stability properties that are defined in terms of a metric are not topological properties in the sense that such properties are not preserved under homeomorphisms of the state space. This is because, in general, homeomorphisms do not preserve a given metric. In contrast, boundedness is a topological property that is preserved under homeomorphisms. This suggests that boundedness may not be related to stability properties that depend on a metric. Indeed, the x_1-axis is Lyapunov stable and attractive in the sense of the Euclidean metric for the planar system $\dot{x}_1 = x_2, \dot{x}_2 = -x_2^3$, since the coordinate x_2, which represents distance from the x_1-axis, decreases monotonically along every nonequilibrium trajectory. However, a simple differentiation along the solutions shows that the orbit of every nonequilibrium point is a level set of the function $x \mapsto x_1 - x_2^{-1}$, and is hence unbounded.

The considerations described above naturally lead us to consider topological notions of stability and attractivity. Our treatment is related to the insightful but short paper [14], which adopts a generalized approach to compare topological as well as metrical notions of stability. While the focus of this paper is the relationship between boundedness and stability, some of our results are related to results stated without proof in [14]. Specific connections with [14] will be pointed out in subsequent sections.

We begin in Section 2 by introducing the necessary preliminaries as well as three progressively stronger notions of boundedness, namely weak boundedness, boundedness, and equi-boundedness. In Section 3, we characterize these notions of boundedness in terms of properties of positive limit sets and the restricted prolongations introduced in [5].

To relate boundedness to stability, we introduce the notion of weak stability of a closed set in Section 4, and show that a set is weakly stable if and only if it contains the restricted prolongation of every point contained in it. Weak stability, which is weaker than Lyapunov stability as well as previous notions of stability for closed sets, is equivalent to Lyapunov stability in the case of compact sets. In fact, we show that a set having an empty interior is Lyapunov stable if and only if it is weakly stable and every point in it has a neighborhood of points whose orbits form a bounded union. Moreover, as we show in Section 7, weak stability has a convenient characterization in terms of Lyapunov functions; the lowest level set of a Lyapunov function that is bounded below and nonincreasing along the trajectories is weakly stable even in the absence of compactness.

In Section 5, we define weak attractivity and attractivity of a closed set, and characterize both in terms of positive limit sets. In particular, we show that the domain of attraction of a closed invariant set $\mathcal{K}$ consists of points whose orbits are bounded and whose positive limit sets are contained in $\mathcal{K}$, thus providing a fundamental link between boundedness and attractivity. We introduce an intermediate notion of attractivity called almost attractivity, which lies between weak attractivity and attractivity in general and is equivalent to attractivity in the case of compact sets. We show that a closed invariant set is attractive if and only if it is almost attractive and every orbit in its domain of almost attraction is bounded. We also show that a Lyapunov stable and weakly attractive closed set is attractive, and its domain of attraction provides an estimate of the set of states having bounded orbits.

In Section 6, we apply the notion of nontangency introduced in [5] to the problem of boundedness. The key result that enables this application is that the restricted prolongation of a point is compact if and only if the vector field describing the dynamics is essentially nontangent to the restricted prolongation of the point, that is, the closure of the set of points at which the vector field fails to be nontangent to the restricted prolongation has compact connected components. A similar result applies to positive limit sets as well. As an application, we show that under the assumption of nontangency, weak stability implies Lyapunov stability, while almost attractivity implies attractivity.

The results of Sections 4–6 are combined in Section 7 to yield our main nontangency-based Lyapunov result for stability and boundedness. Our result states that the lowest level set of a Lyapunov function that is bounded below and nonincreasing along the dynamics is weakly stable, while all weakly bounded solutions are almost attracted to the zero level set of the Lyapunov derivative. In addition, if the dynamics are essentially nontangent to the lowest level set, then the lowest level set is Lyapunov stable and has a neighborhood of points having bounded orbits. Under an additional condition, the lowest level set is attractive, and its domain of attraction provides an inner estimate of the set of points having bounded orbits. In the case where the lowest level set is compact, essential nontangency holds trivially, and our result reduces to the well-known theorems of Lyapunov and LaSalle for stability and asymptotic stability, respectively. We provide an example to illustrate the application of our main result. Examples are also provided to illustrate the notions of boundedness, stability and attractivity introduced in Sections 3–5.

2. Preliminaries

The notions of openness, convergence, continuity and compactness that we use refer to the usual topology on $\mathbb{R}^n$.

Given $\mathcal{K} \subseteq \mathbb{R}^n$, we let int$\mathcal{K}$, bd $\mathcal{K}$ and $\overline{\mathcal{K}}$ denote the interior, boundary, and closure, respectively, of $\mathcal{K}$. An open neighborhood of $\mathcal{K}$ is an open set containing $\mathcal{K}$. A closed set $\mathcal{K} \subseteq \mathbb{R}^n$ equals the intersection of every sequence $\{\mathcal{U}_i\}$ of open neighborhoods of $\mathcal{K}$ satisfying $\overline{\mathcal{U}}_{i+1} \subset \mathcal{U}_i$. A set $\mathcal{U} \subseteq \mathbb{R}^n$ is *bounded* if $\overline{\mathcal{U}}$ is compact. A point $x \in \mathbb{R}^n$ is a *subsequential*