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Large deviations estimates for some non-local equations Fast decaying kernels and explicit bounds

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ABSTRACT

We study large deviations for some non-local parabolic type equations. We show that, under some assumptions on the non-local term, problems defined in a bounded domain converge with an exponential rate to the solution of the problem defined in the whole space. We compute this rate in different examples, with different kernels defining the non-local term, and it turns out that the estimate of convergence depends strongly on the decay at infinity of that kernel.

1. Introduction

Consider continuous and bounded solutions $u : \mathbb{R}^N \times [0, \infty) \to \mathbb{R}$ of the linear non-local equation

$$\frac{\partial u}{\partial t}(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,\mathrm{d}y - u(x,t) \quad \text{in } \mathbb{R}^N \times (0,\infty), \tag{1.1}$$

where $u(x, 0) = u_0(x)$ is fixed, bounded and continuous. For simplicity, we will assume throughout the paper that solutions are non-negative, and thus also $u_0 \geqslant 0$. The kernel J is assumed to be a symmetric, continuous probability density.

The main contribution of this paper is to describe how solutions u_R of (1.1), but defined in the ball $B_R = \{x \in \mathbb{R}^N : |x| \leqslant R\}$, converge to the solution u of (1.1).

Let us first explain what u_R is exactly: we consider here the notion of Dirichlet problem that consists in putting $u_R = 0$ not only on the topological boundary of B_R, but also in all the complement of $B_R \times [0, \infty)$. In this way, u_R solves the equation

$$\frac{\partial u_R}{\partial t}(x,t) = \int_{B_R} J(x-y)u_R(y,t)\,\mathrm{d}y - u_R(x,t) \quad \text{in } B_R \times (0,\infty) \tag{1.2}$$

with initial data $u_R(x, 0) = u_0(x)$ in B_R. We refer to [1] and [2] for more information on these non-local Dirichlet problems, and also [3] for similar questions (with singular kernels).

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As one can imagine, under suitable assumptions, u_R will reasonably converge to u as $R \to \infty$, but we want to obtain some estimates of how fast convergence occurs. Actually, this problem may be seen as a numerical question since of course, computing numerically solutions requires a bounded domain. In this case one has to know how far from the real solution the computed one is.

In the case of the Heat Equation, the answer is given in [4]: the distance between u and u_R in the ball of radius θR (with $0 < \theta < 1$) is estimated by

$$\sup_{|x|\leqslant\theta R}(u-u_R)\leqslant \exp\left(-R^2\frac{(1-\theta)^2}{4t}+o(1)\right),$$

which means that convergence occurs exponentially fast, with a rate of the order of R^2 inside the exponential.

Our aim is to produce similar estimates for non-local equations (1.1). We face here several difficulties, which imply nontrivial adaptations of ideas and techniques in [4], that we list below:

(i) Various behaviours of J imply various rates: the importance of the tail of J enters into play, since the operator puts emphasis on the difference between u and $u_R = 0$ far from the point where we compute u_R, as we shall explain below in the subsection devoted to the probabilistic aspects. Roughly speaking, the more J is big at infinity, the slower u_R converges to u.

(ii) The structure of the Hamiltonian which describes the rate function $\mathcal{I}$ is completely different: for the Heat Equation the associated Hamilton–Jacobi equation is $\partial_t \mathcal{I} + |\nabla \mathcal{I}|^2 = 0$, hence $H(p) = p^2$. Here the problem for the rate function is related to the Hamiltonian

$$H(p)=\int_{\mathbb{R}^N}\left(e^{p\cdot y}-1\right)J(y)\,dy, \tag{1.3}$$

which, although it is local, is the limit as $R \to \infty$ of Hamiltonians involving a non-local term,

$$H(x,v,p,M,\mathcal{L}_R[v])=-\int_{\mathbb{R}^N}\left(e^{-R\{v(x+y/R)-v(x)\}}-1\right)J(y)\,dy=-\mathcal{L}_R[v].$$

This localization process is one of the main interesting features of this problem.

(iii) We are not facing here a diffusive effect, but more a transport effect: in the case of the Heat equation, the scaling used in [4] in order to prove convergence is the parabolic (Rx, t) one (equivalent to (R^2x, Rt)). Here, we have to use a hyperbolic change of variables (Rx, Rt) in order to scale the problem in a suitable way.

Probabilistic context — The term "large deviation" comes from the French "grands écarts" which was used first to describe how far from the normal distribution, some exceptional events are. For instance, it is well known that if (X_n) is a sequence of independent and identically distributed random variables with $\mathbb{E}(X_i) = \mu$, then

$$\frac{X_1+X_2+\cdots+X_n}{n}\xrightarrow{\mathbb{P}}\mu$$

as $n \to +\infty$, the convergence occurring in law (this is the law of large numbers). Now, one may wonder how to estimate, for $\varepsilon > 0$ small, the quantity:

$$\mathbb{P}\left(\left|\frac{X_1+X_2+\cdots+X_n}{n}-\mu\right|>\varepsilon\right).$$

A result of Cramer (1938, see [5] for a proof), shows that if one defines the rate function

$$I(\varepsilon):=\lim_{n\to+\infty}-\frac{1}{n}\log\mathbb{P}\left(\left|\frac{X_1+X_2+\cdots+X_n}{n}-\mu\right|>\varepsilon\right),$$

then

$$I(\varepsilon)=\sup_{t\in\mathbb{R}}\{\varepsilon t-\log M(t)\},\quad \text{where } M(t)=\mathbb{E}[e^{tX_1}]<\infty,$$

which implies,

$$\mathbb{P}\left(\left|\frac{X_1+X_2+\cdots+X_n}{n}-\mu\right|>\varepsilon\right)\leqslant e^{-nI(\varepsilon)}.$$

This exponential behaviour is typical of what is called "large deviations".

In this paper, $(u-u_R)(\cdot, T)$ measures in some sense the total amount of process that can escape from the ball B_R between times 0 and T (see [4] and [1] for more explanations about this aspect). Thus, our results may be viewed as "large deviations" results in the sense that the probability of escaping the ball B_R up to a given time becomes small as $R \to \infty$. Exponentially small in fact, with a rate which depends on the tail of J since this tail measures the amount of "big jumps". Values of J near the origin only concern "small jumps" that are not relevant as far as escaping the ball is at stake. So that is why, as we explain in Section 5, adding a singularity at the origin does not change the rate of convergence.