Nonlinear Analysis 71 (2009) 5688–5694

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

$\mathcal{T}$-class algorithms for pseudocontractions and κ-strict pseudocontractions in Hilbert spaces

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A R T I C L E   I N F O

Article history:
Received 2 June 2008
Accepted 29 April 2009

Keywords:
Nonexpansive mappings
Viscosity approximation
Fixed points

A B S T R A C T

In this paper we study iterative algorithms for finding a common element of the set of fixed points of κ-strict pseudocontractions or finding a solution of a variational inequality problem for a monotone, Lipschitz continuous mapping. The last problem being related to finding fixed points of pseudocontractions. These algorithms were already studied in [G.L. Acedo, H.-K. Xu, Iterative methods for strict pseudo-contractions in hilbert spaces, Nonlinear Analysis 67 (2007) 2258–2271] and [N. Nadezhkina, W. Takahashi, Strong convergence theorem by a hybrid method for nonexpansive mappings and lipschitz-continuous monotone mappings, SIAM Journal on Optimization 16 (4) (2006) 1230–1241] but our aim here is to provide the links between these known algorithms and the general framework of $\mathcal{T}$-class algorithms studied in [H.H. Bauschke, P.L. Combettes, A weak-to-strong convergence principle for fejér-monotone methods in hilbert spaces, Mathematics of Operations Research 26 (2) (2001) 248–264].

1. Introduction

Let C be a closed convex subset of a Hilbert space $\mathcal{H}$ and P_C be the metric projection from $\mathcal{H}$ onto C. A mapping $Q : C \mapsto C$ is said to be a *strict pseudocontraction* if there exists a constant $0 \leq \kappa < 1$ such that:

$$\|Qx - Qy\|^2 \leq \|x - y\|^2 + \kappa \|(I - Q)x - (I - Q)y\|^2, \tag{1}$$

for all $x, y \in C$. A mapping Q for which (1) holds is also called a κ-strict pseudocontraction. As pointed out in [1], iterative methods for finding a common element of the set of fixed points of strict pseudocontractions are far less developed than iterative methods for nonexpansive mappings ($\kappa = 0$). Our main goal in this paper is to show that two specific algorithms studied in [1] and [2] can be linked to a general class of algorithms, called $\mathcal{T}$-class algorithms, and previously studied in [3]. For that purpose we slightly extend the $\mathcal{T}$-class definition. Thus, we do not provide new algorithms and new convergence theorems, but extend the scope of algorithms covered by the $\mathcal{T}$-class framework. Section 2 is devoted to the case $0 \leq \kappa < 1$ and considers Algorithm 1 studied in [1] rephrased as a $\mathcal{T}$-class algorithm. Section 3 is devoted to the case $\kappa = 1$ for which the previous algorithm cannot be used. A mapping A for which (1) holds with $\kappa = 1$ is called *pseudocontractive*. We will see that *pseudocontractive* mappings are related to monotone Lipschitz continuous mappings. A mapping $A : C \mapsto \mathcal{H}$ is called *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0 \quad \text{for all } (u, v) \in C^2.$$

A is called k-Lipschitz continuous if there exists a positive real number k such that

$$\|Au - Av\| \leq k\|u - v\| \quad \text{for all } (u, v) \in C^2.$$

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doi:10.1016/j.na.2009.04.049

Let the mapping $A : C \mapsto \mathcal{H}$ be monotone and Lipschitz continuous. The variational inequality problem is to find a $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0 \quad \text{for all } v \in C.$$

The set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

Assume that a mapping $Q : C \mapsto C$ is pseudocontractive and k-Lipschitz continuous. Then, the mapping $A = I - Q$ is monotone and $(k + 1)$-Lipschitz continuous and moreover $Fix(Q) = VI(C, A)$ [2, Theorem 4.5], where $Fix(Q)$ is the set of fixed points of Q. That is

$$Fix(Q) \stackrel{\text{def}}{=} \{x \in C : Qx = x\}. \tag{2}$$

Thus, to cover the case $\kappa = 1$, we investigate algorithms which aim at computing $P_{VI(C,A)}x$ for a monotone and k-Lipschitz continuous mapping A. We will, in Section 3, mainly use results from [2] to prove that the general algorithm used in [2] can be rephrased in a slightly extended $\mathcal{T}$-class algorithm framework.

2. $\mathcal{T}$-class iterative algorithm for a sequence of κ-strict pseudocontractions

Let $(Q_n)_{n\geq 0}$ be a sequence of κ-strict pseudocontractions for $\kappa \in [0, 1)$ and $(\alpha_n)_{n\geq 0}$ a sequence of real numbers chosen so that $\alpha_n \in (\kappa, 1)$. We consider, as in [1], the following algorithm:

Algorithm 1. Given $x_0 \in C$, we consider the sequence $(x_n)_{n\geq 0}$ generated by the following algorithm:

$$\begin{aligned}
&y_n = \alpha_n x_n + (1 - \alpha_n)Q_n x_n,\\
&C_n \stackrel{\text{def}}{=} \left\{z \in C \mid \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \kappa)\|x_n - Q_n x_n\|^2\right\},\\
&D_n \stackrel{\text{def}}{=} \{z \in C \mid \langle x_n - z, x_0 - x_n\rangle \geq 0\},\\
&x_{n+1} = P_{(C_n\cap D_n)}x_0.
\end{aligned}$$

We will show that this algorithm belongs to the $\mathcal{T}$-class algorithms as defined in [3] and deduce its strong convergence to $P_F x_0$ when $F \neq \emptyset$ and where $F \stackrel{\text{def}}{=} \cap_{n\geq 0} Fix(Q_n)$.

For $(x, y) \in \mathcal{H}^2$, define the mapping H as follows:

$$H(x, y) \stackrel{\text{def}}{=} \{z \in \mathcal{H} \mid \langle z - y, x - y\rangle \leq 0\} \tag{3}$$

and denote by $Q(x, y, z)$ the projection of x onto $H(x, y) \cap H(y, z)$. Note that $H(x, x) = \mathcal{H}$ and for $x \neq y$, $H(x, y)$ is a closed affine half space onto which y is the projection of x.

Lemma 2. *The sequence generated by Algorithm 1 coincides with the sequence given by $x_{n+1} = Q(x_0, x_n, T_n x_n)$ with:*

$$T_n(x) \stackrel{\text{def}}{=} \frac{x + R_n x}{2} + \frac{1}{2}\left(\frac{\kappa - \alpha_n}{1 - \alpha_n}\right)(x - R_n x), \quad \text{and} \quad R_n(x) \stackrel{\text{def}}{=} \alpha_n x + (1 - \alpha_n)Q_n(x). \tag{4}$$

Moreover, we have:

$$2T_n - I = \kappa I + (1 - \kappa)Q_n x. \tag{5}$$

Proof. Let $\kappa \in [0, 1)$, $\alpha \in (\kappa, 1)$, $y \stackrel{\text{def}}{=} \alpha x + (1 - \alpha)Qx$ for κ-strict pseudocontraction Q and define $\Gamma(x, y)$ as follows:

$$\Gamma(x, y) \stackrel{\text{def}}{=} \left\{z \in \mathcal{H} \mid \|y - z\|^2 \leq \|x - z\|^2 - (1 - \alpha)(\alpha - \kappa)\|x - Qx\|^2\right\}. \tag{6}$$

We first prove that $\Gamma(x, y) = H(x, Tx)$ where T is defined by Eq. (4).

$$\begin{aligned}
&\|y - z\|^2 - \|x - z\|^2 \leq -(1 - \alpha)(\alpha - \kappa)\|x - Qx\|^2\\
&\quad\Leftrightarrow \langle y - z, y - z\rangle - \|x - z\|^2 \leq -(1 - \alpha)(\alpha - \kappa)\|x - Qx\|^2\\
&\quad\Leftrightarrow \langle y - x, y - z\rangle + \langle x - z, y - z\rangle - \|x - z\|^2 \leq -(1 - \alpha)(\alpha - \kappa)\|x - Qx\|^2\\
&\quad\Leftrightarrow \langle y - x, y - z\rangle + \langle x - z, y - x\rangle \leq -(1 - \alpha)(\alpha - \kappa)\|x - Qx\|^2\\
&\quad\Leftrightarrow \langle y - x, y - z\rangle + \langle x - z, y - x\rangle \leq (\alpha - \kappa)\,\langle y - x, x - Qx\rangle\\
&\quad\Leftrightarrow \langle y - x, y + x - 2z + (\kappa - \alpha)(x - Qx)\rangle \leq 0\\
&\quad\Leftrightarrow \left\langle y - x, y + x - 2z + \left(\frac{\kappa - \alpha}{1 - \alpha}\right)(x - y)\right\rangle \leq 0
\end{aligned}$$