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A note on distributional chaos with respect to a sequence

Piotr Oprocha

Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland

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ABSTRACT

The aim of this note is to use methods developed by Kuratowski and Mycielski to prove that some more common notions in topological dynamics imply distributional chaos with respect to a sequence. In particular, we show that the notion of distributional chaos with respect to a sequence is only slightly stronger than the definition of chaos due to Li and Yorke. Namely, positive topological entropy and weak mixing both imply distributional chaos with respect to a sequence, which is not the case for distributional chaos as introduced by Schweizer and Smítal.

1. Introduction

In a recent paper [1], the authors introduced a generalized version of distributional chaos. Let $\{p_i\}_{i\in\mathbb{N}}$ be an increasing sequence, (X, d). For any positive integer n, points $x, y \in X$ and $t \in \mathbb{R}$, let

$$\Phi_{xy}^{(n)}(t, \{p_i\}_{i\in\mathbb{N}}) := \frac{1}{n}\left|\left\{i : d(f^{p_i}(x), f^{p_i}(y)) < t, 0 \le i < n\right\}\right|,$$

$$\Phi_{xy}(t, \{p_i\}_{i\in\mathbb{N}}) := \liminf_{n\to\infty} \Phi_{xy}^{(n)}(t, \{p_i\}_{i\in\mathbb{N}}),$$

$$\Phi_{xy}^{*}(t, \{p_i\}_{i\in\mathbb{N}}) := \limsup_{n\to\infty} \Phi_{xy}^{(n)}(t, \{p_i\}_{i\in\mathbb{N}}),$$

where $|A|$ denotes the cardinality of a set A. Using these notations, distributional chaos with respect to a sequence is defined as follows:

Definition 1. A pair of points $(x, y) \in X \times X$ is called *distributionally chaotic with respect to a sequence* $\{p_i\}_{i\in\mathbb{N}}$, if $\Phi_{xy}(s, \{p_i\}_{i\in\mathbb{N}}) = 0$ for some $s > 0$ and $\Phi_{xy}^{*}(t, \{p_i\}_{i\in\mathbb{N}}) = 1$ for all $t > 0$.

A set S containing at least two points is called *distributionally scrambled with respect to* $\{p_i\}_{i\in\mathbb{N}}$, if any pair of distinct points of S is distributionally chaotic with respect to $\{p_i\}_{i\in\mathbb{N}}$.

A map f is *distributionally chaotic with respect to* $\{p_i\}_{i\in\mathbb{N}}$, if it has an uncountable set distributionally scrambled with respect to $\{p_i\}_{i\in\mathbb{N}}$.

These definitions generalize the well-known notion of distributional chaos introduced in [2]. Namely, a pair $(x, y) \in X \times X$ (or a map f) is distributionally chaotic of type 1 [3], if it is distributionally chaotic with respect to $\mathbb{N}$. Note, that distributional chaos is also a generalization of chaos in the sense of Li and Yorke [4] (see [5] for a very nice survey on this kind of chaos).

E-mail address: oprocha@agh.edu.pl.

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A main result of [1] is the following:

Theorem 2. *Let $f : X \to X$ be continuous and let $h : X \to \Sigma_2$ be a factor map (i.e. h is continuous, onto and $h \circ f = \sigma \circ h$, where (Σ_2, σ) is the full shift over two letters alphabet). In that case there exists an increasing sequence of positive integers $\{p_i\}_{i\in\mathbb{N}}$, such that f exhibits distributional chaos with respect to the sequence $\{p_i\}_{i\in\mathbb{N}}$.*

In view of another theorem from [1] presented below, it appears that Theorem 2 can be extended:

Theorem 3. *Let f be a continuous map from the interval $[0, 1]$ into itself. Then map f is chaotic in the sense of Li and Yorke iff it is distributionally chaotic with respect to some increasing sequence $\{p_i\}_{i\in\mathbb{N}}$.*

Intuitively, Definition 1 is very similar to the definition of chaos in the sense of Li and Yorke, so we may hope that it shares similar properties. Recently, it was proven that chaos in the sense of Li and Yorke is present in systems with positive topological entropy [6]. In particular, this is the case for extensions of full shifts over two or more symbols. The main aim of this article is to prove that our intuition about topological entropy and distributional chaos (with respect to a sequence) is correct, and that an even stronger property holds. Before we formulate the main results, let us recall the scientific background of our approach.

One of the first attempts to use properties of residual relations in a construction of a scrambled set was made by Iwanik [7] during the proof that weak mixing implies chaos in the sense of Li and Yorke. A similar method was applied in [8], where some results of Kuratowski [9] on independent sets were extended. Another result of this kind (without direct reference to Kuratowski) was the proof that a transitive map with at least one fixed point has an uncountable and dense scrambled set (that does not have to be σ-Cantor) [10]. Moreover, in [6] it was proven that positive topological entropy implies chaos in the sense of Li and Yorke (we will recall this result during the development of this article). Later, it was observed that the approaches of Mycielski [11] and Kuratowski [9] contain very strong tools even in a wider context. Their methods and results have been collected, extended and presented in a more recent way in a monograph by Akin [12]. The most important tools used in our article are hereditary sets and their properties [12].

Let $f^{\times k}$ denote the k-times Cartesian product of f and let $\Delta_X^{(k)} := \left\{(x, \dots, x) \in X^k \;:\; x \in X\right\}$ be the diagonal in X^k. Then, the main result of our paper is the following theorem:

Theorem 4. *Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous onto map. Let $W^{(k)} \subset X^k$ be a closed $f^{\times k}$-invariant subset such that $\Delta_X^{(k)} \subset W^{(k)}$ and $f^{\times k}|_{W^{(k)}}$ is transitive. If there is a perfect set $S \subset X$ such that $W^{(k)} \cap S^k$ is residual in S^k for every $k = 2, 3, \dots$, then there is an increasing sequence $\{p_i\}_{i\in\mathbb{N}}$ and a Mycielski set $M \subset S$ dense in S with the property that M is distributionally scrambled with respect to the sequence $\{p_i\}_{i\in\mathbb{N}}$.*

From this theorem, we obtain two corollaries:

Corollary 5. *Let (X, d) be a compact metric space and let $f : X \to X$ be a continuous onto map. If f has positive topological entropy, then there is a perfect and invariant set Z such that for every open set U, $U \cap Z \neq \emptyset$, there is a Cantor set $C \subset U \cap Z$ and an increasing sequence $\{p_i\}_{i\in\mathbb{N}}$, such that C is distributionally scrambled with respect to the sequence $\{p_i\}_{i\in\mathbb{N}}$.*

Corollary 6. *Let (X, d) be a compact metric space without isolated points and let $f : X \to X$ be a continuous onto map. If f is weakly mixing, then there is a dense Mycielski set $M = \bigcup_{i=1}^{\infty} C_i$ with the following property: For every integer $n > 0$, there is an increasing sequence $\{p_i\}_{i\in\mathbb{N}}$ such that the Cantor set $\hat{C}_i = \bigcup_{i=1}^{n} C_i$ is distributionally scrambled with respect to the sequence $\{p_i\}_{i\in\mathbb{N}}$.*

Remark 7. There are examples of a weakly mixing map or a map with positive topological entropy which are not distributionally chaotic of type 1 (see [13,14] respectively). This proves that the notion of distributional chaos with respect to a sequence is really a weaker condition.

All notations missing up to this point will be introduced in the next section.

2. Basic definitions and notations

Let (X, d) be a compact metric space. A set $A \subset X$ is *perfect* if it is closed and has no isolated points. If, in addition, A is totally disconnected (i.e. the only connected subsets of A are singletons), then we say that A is a Cantor set. A set $M = \bigcup_{i=1}^{\infty} C_i$ where each C_i is a Cantor set, is called a *σ-Cantor set* or a *Mycielski set*. A set $R \subset X$ is G_δ, if it is at most countable intersection of open sets; it is *residual*, if it contains a dense G_δ-set.

We shall denote the set $\left\{x, f(x), f^2(x), \dots\right\}$ by $\mathrm{Orb}^+(x)$ and call it the *positive orbit* of a point x. A point $y \in X$ is an *ω-limit point* of a point x, if it is an accumulation point of the sequence $x, f(x), f^2(x), \dots$. The set of all ω-limit points of x is called *ω-limit set* of x and denoted by $\omega(x, f)$. We say that a point x is *periodic*, if $f^n(x) = x$ for some $n \geq 1$, and *recurrent*, if $x \in \omega(x, f)$.