Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis 69 (2008) 2379–2392
Nonlinear Analysis
www.elsevier.com/locate/na

Existence and continuous dependence results for nonlinear differential inclusions with infinite delay☆

Zhenbin Fan

Yangzhou Polytechnic College, Yangzhou, Jiangsu, 225002, PR China

Received 22 July 2007; accepted 7 August 2007

Abstract

This paper is concerned with nonlinear functional differential inclusions with infinite delay in Banach spaces. Using tools involving the measure of noncompactness and multi-valued fixed point theory, existence and continuous dependence results are obtained, for integral solutions, without the assumption of compactness on the associated nonlinear semigroup.

MSC: 34A60; 47H06; 47H20

Keywords: Measure of noncompactness; Infinite delay; Phase space; Fixed point; Differential inclusion; Accretive operator; Integral solution

1. Introduction

The study of functional differential equations with infinite delay in an abstract admissible phase space was initiated by Hale and Kato [12] and Schumacher [24] (for $X = R_n$). Subsequently, many authors are devoted to the study of this field because the admissible phase space is adopted for equations with delay and it is useful to solve a lot of practical problems in applications. For more details, we refer to the work [4,6,8,10–14,16–24] and references therein.

In [17], Kartsatos and Parrott studied the following Cauchy problem for the nonlinear functional differential equation

$$u'(t) + A(t)u(t) \ni G(t, u_t), \quad 0 < t \leq T, \tag{1.1a}$$
$$u(t) = \varphi(t), \quad t \in I, \tag{1.1b}$$

where X is a general Banach space, $A(t) : D(A(t)) \subset X \to X$ is m-accretive for every $t \in [0, T]$, G is Lipschitz continuous in both variables, and $\varphi \in C(I; X)$, $I = [-r, 0]$, for some fixed positive constant r, or $I = (-\infty, 0]$. They gave the existence results of weak solutions to the problem (1.1) by a fixed point argument. In [18], Kartsatos and Shin studied the problem (1.1) to prove the existence of local solutions by the Schauder's fixed point theorem in $C(0, T; X)$ as in [22] with the continuity of G and $\varphi \in C(I; X)$, $I = [-r, 0]$, for some fixed positive constant r. In [10,14], the authors considered the similar problem by using the Schauder's fixed point theorem in $L^p(0, T; X)$ for

☆ This work was supported by the PRC grant NSFC 10571150.
E-mail address: fzbmath@yahoo.com.cn.

0362-546X/$ - see front matter
doi:10.1016/j.na.2007.08.011

$1 \leq p \leq \infty$ with the Lipschitz continuity of G and $\varphi \in L^p(-r, 0; X)$. However, the methods in [10,14,18] cannot be applied directly to the case $I = (-\infty, 0]$.

In [8], Gori, Obukhovskii, Ragin and Rubbioni discussed the following initial problem for systems governed by a semilinear functional differential inclusion

$$u'(t) \in Au(t) + F(t, u_t), \quad t \geq \sigma, \tag{1.2a}$$

$$u_\sigma = \varphi \in \mathscr{B}, \tag{1.2b}$$

where A is the infinitesimal generator of a strongly continuous semigroup of linear operators, F is a multi-valued function and $\mathscr{B}$ is a phase space satisfying a set of axioms. They obtained the existence results of mild solutions to (1.2) by using the multi-valued condensing fixed point techniques. As to the study of other semilinear differential equations with delay in Banach spaces, we refer readers to the recent works [4,6,7,19–21].

Motivated by the above approach, the goal in the present paper is to use the multi-valued fixed point arguments to obtain the integral solutions of the following nonlinear differential inclusion with infinite delay

$$u'(t) \in -Au(t) + F(t, u_t), \quad \sigma < t \leq \sigma + T, \tag{1.3a}$$

$$u_\sigma = \varphi \in \mathscr{B}, \tag{1.3b}$$

in a real Banach space X. Here we assume that A is an m-accretive operator such that $-A$ generates an equicontinuous semigroup, F is a multifunction, $\mathscr{B}$ is a phase space and X^* is uniformly convex. In our proof, we do not need the compactness of the semigroup. Our basic tools are the methods and results for nonlinear differential equations, the properties of noncompact measures and multi-valued fixed point techniques. Our results extend the ones in [10,14,18] to the case of infinite delay and the ones in [8] to the fully nonlinear case. As a matter of fact, it should be mentioned that our main results here are nontrivial nonlinear extensions of those in [8], and our proofs herein are essentially new even in the single valued case.

The outline of this paper is as follows. In Section 2, we recall some facts about the measure of noncompactness and multi-valued mappings. In Section 3, we define the integral solution operator and give its properties. Section 4 presents the existence of local integral solutions for problem (1.3). We prove the existence of global integral solutions and obtain the compactness of the solution set in Section 5. Finally, the continuous dependence results about the integral solution set are obtained.

2. Preliminaries

Throughout this paper, σ is a real number and $T > 0$. $(X, \|\cdot\|)$ is a real Banach space with dual X^*, R^+ stands for the set of nonnegative numbers, and we denote by $P(X)$, $P_{kv}(X)$ the collection of all nonempty subsets of X, all nonempty compact convex subsets of X, respectively, and by $C(\sigma, \sigma + T; X)$, $L^1(\sigma, \sigma + T; X)$ the space of all continuous functions, all Bochner summable functions, respectively. For any function $y : (-\infty, \sigma + T] \to X$ and for every $t \in (-\infty, \sigma + T]$, y_t represents the function from $(-\infty, 0]$ into X defined by $y_t(\theta) = y(t + \theta)$, $-\infty < \theta \leq 0$.

We recall the following definitions.

Definition 2.1. A linear topological space of functions from $(-\infty, 0]$ into X, with seminorm $\|\cdot\|_{\mathscr{B}}$, is called an admissible phase space if $\mathscr{B}$ has the following properties:

If $y : (-\infty, \sigma + T] \to X$ is continuous on $[\sigma, \sigma + T]$ and $y_\sigma \in \mathscr{B}$, then for every $t \in [\sigma, \sigma + T]$, we have

(B1) $y_t \in \mathscr{B}$;

(B2) the function $t \to y_t$ is continuous;

(B3) $\|y_t\|_{\mathscr{B}} \leq K(t - \sigma) \sup_{\sigma \leq \tau \leq t} \|y(\tau)\| + M(t - \sigma)\|y_\sigma\|_{\mathscr{B}}$,

where $K, M : [0, +\infty) \to [0, +\infty)$ are independent of y, K is strictly positive and continuous, and M is locally bounded.

Definition 2.2. Let E^+ be the positive cone of an order Banach space $(E, \leq)$. A function Φ defined on the set of all bounded subsets of the Banach space X with values in E^+ is called a measure of noncompactness (MNC) on X if $\Phi(\overline{\text{co}}\Omega) = \Phi(\Omega)$ for all bounded subsets $\Omega \subset X$, where $\overline{\text{co}}\Omega$ stands for the closed convex hull of Ω.