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Partial functional differential equation with an integral condition and applications to population dynamics

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Abstract

In this work we consider a semilinear functional partial differential equation with an integral condition. We apply the method of semidiscretization in time, to establish the existence and uniqueness of solutions. We also study the continuation of the solution to the maximal interval of existence. Finally we give examples to demonstrate the applications of our results.

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1. Introduction

Let $\mathbf{H} = L^2(0, 1)$ be the real Hilbert space of all real square integrable functions on $(0, 1)$ with the standard inner product and the corresponding norm generated by the inner product denoted by $(.,)$ and $\|\cdot\|$, respectively. We identify a function $u : (0, 1) \times [-\tau, T] \to \mathbb{R}$, $\tau > 0$, such that for each $t \in [-\tau, T]$, $u(., t) \in \mathbf{H}$ with the function $u : [-\tau, T] \to \mathbf{H}$ by the rule $u(t)(x) = u(x, t)$, $t \in [-\tau, T]$, $x \in (0, 1)$. The Hilbert space $\mathbf{H}$ may be embedded in another Hilbert space $\mathbf{B}_2^1(0, 1)$ [3,4] which is the completion of $C_0(0, 1)$, the space of all continuous functions on $(0, 1)$ having compact support in $(0, 1)$, with the inner product

$$(u, v)_{B_2^1} = \int_0^1 \left(\int_0^x u(\xi)\mathrm{d}\xi \int_0^x v(\xi)\mathrm{d}\xi \right) \mathrm{d}x$$

and the corresponding norm $\|u\|^2_{B_2^1} = (u, u)_{B_2^1}$. It follows that $\|u\|_{B_2^1} \leq \frac{1}{2}\|u\|$. Let $0 < T < \infty$ and for $t \in [0, T]$, let $\mathbf{C}_t := C([-\tau, t]; \mathbf{B}_2^1(0, 1))$ be the space of all continuous functions from $[-\tau, t]$ into $\mathbf{B}_2^1(0, 1)$ endowed with the norm

$$\|\chi\|_t := \sup_{-\tau \leq s \leq t} \|\chi(s)\|_{B_2^1}, \quad \chi \in \mathbf{C}_t.$$

For $\chi \in \mathbf{C}_T$, we denote $\chi_t \in \mathbf{C}_0$ given by $\chi_t(\theta) = \chi(t + \theta)$, $\theta \in [-\tau, 0]$.

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In this paper we are concerned with the following semilinear partial functional differential equation with an integral condition,

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t,u(x,t),u_t), \quad (x,t) \in (0,1) \times [0,T], \tag{1}$$

$$u(x,t) = \Phi(x,t), \quad t \in [-\tau, 0],\ x \in (0,1), \tag{2}$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \quad t \in [0,T], \tag{3}$$

$$\int_0^1 u(x,t)\mathrm{d}x = 0, \quad t \in [0,T], \tag{4}$$

where the unknown function $u : [-\tau, T] \to \mathbf{H}$ is to be sought such that $u \in \mathbf{C}_T$, the map f is defined from $(0,1) \times [0,T] \times \mathbb{R} \times \mathbf{D}$ into $\mathbb{R}$, with a suitably chosen $\mathbf{D} \subset \mathbf{C}_0$, and the history function $\Phi : [-\tau, 0] \to \mathbf{H}$ with $\Phi \in \mathbf{C}_T$.

Our aim is to apply the method of semidiscretization in time, also known as the method of lines or Rothe's method, to establish the existence, uniqueness of a solution and the unique continuation of a solution to the maximal interval of existence. We note that there is no loss of generality in considering the homogeneous conditions in (3) and (4) as the more general problem (1)–(4) with u, f and Φ replaced by v, g, Ψ and conditions (3) and (4) replaced by

$$\frac{\partial v}{\partial x}(0,t) = U(t), \quad t \in [0,T], \tag{5}$$

$$\int_0^1 v(x,t)\mathrm{d}x = V(t), \quad t \in [0,T], \tag{6}$$

respectively, where U and V are differentiable functions on $[0,T]$, may be reduced to (1)–(4) using the transformations

$$u(x,t) = v(x,t) - \left(x - \frac{1}{2}\right) U(t) - V(t)$$

and

$$f(x,t,r,\chi) = g\left(x, t, r + \left(x - \frac{1}{2}\right) U(t) - V(t), \chi + \left(x - \frac{1}{2}\right) U_t - V_t\right) - \left(x - \frac{1}{2}\right) \frac{\mathrm{d}U(t)}{\mathrm{d}t} - \frac{\mathrm{d}V(t)}{\mathrm{d}t},$$

$$\tilde{\Phi} = \Phi - \left(x - \frac{1}{2}\right) U_t - V_t,$$

with $U_t, V_t \in \mathbf{C}_0$ given by $U_t(\theta) = U(t+\theta)$, $V_t(\theta) = V(t+\theta)$ such that $U(t+\theta) = U(0)$ and $V(t+\theta) = V(0)$ for $t + \theta \le 0, \theta \in [-\tau, 0]$.

The initial work on heat equations with integral conditions has been carried out by Cannon [6]. Subsequently, similar studies have been done by Kamynin [9], Ionkin [7]. Beilin [2] has considered the wave equation with an integral condition using the method of separation of variables and Fourier series.

Pulkina [12] has dealt with a hyperbolic problem with two integral conditions and has established the existence and uniqueness of generalized solutions using the fixed point arguments.

Our analysis is motivated by the works of Bouziani and Merazga [10,5] and Bahuguna and Shukla [1]. In [10,5] the authors have used the method of semidiscretization to (1)–(4) without delays. In [1] the method of semigroups of bounded linear operators in a Banach space is used to study a partial differential equation involving delays arising in the population dynamics. We use the method of semidiscretization in time first to establish the local existence of a unique solution of (1)–(4) on a subinterval $[-\tau, T_0]$, $0 < T_0 \le T_0$ and then extend it either to the whole interval $[-\tau, T]$ or to the maximal subinterval $[-\tau, T_{\max}) \subset [-\tau, T]$ of existence with $\lim_{t \to T_{\max}-} \|u(t)\| = +\infty$.