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Quasi-random sampling for signal recovery

Mirosław Pawlak [a,*], Ewaryst Rafajłowicz [b]

[a] Department of Electrical & Computer Engineering, The University of Manitoba, Winnipeg, Manitoba, Canada, R3T 2N2

[b] Institute of Computer Engineering, Control and Robotics Wrocław University of Technology, Wroclaw, Poland

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ABSTRACT

The problem of reconstruction of band-limited signals from nonuniformly sampled and and noisy observations is studied. It is proposed to sample a signal at quasi-random points that form a deterministic sequence, with properties resembling a random variable, being uniformly distributed. Such quasi-random points can be easily and efficiently generated. The reconstruction method based on the modified Whittaker–Shannon cardinal expansion is proposed and its asymptotical properties are examined. In particular, the sufficient conditions for the convergence of the mean integrated squared error are found. The key difference between the proposed reconstruction algorithm and the classical Whittaker–Shannon scheme is in treating the sampling rate and the reconstruction rate differently. This distinction is necessary to ensure consistency of the reconstruction algorithm in the presence of noise.

1. Introduction and preliminaries

Throughout the paper we assume that a signal $f(t)$ has a bounded spectrum, i.e., that its Fourier transform $F(\omega)$ vanishes outside of a finite interval. Any signal with such a property is referred to as band-limited. A fundamental result in signal processing (see [1–4]) is that any band-limited signal $f(t)$ can be perfectly recovered from its discrete values $f(k\frac{\pi}{\Omega})$, $k = 0, \pm 1, \pm 2, \ldots$. In fact, the celebrated Whittaker–Shannon sampling theorem says that

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(k\frac{\pi}{\Omega}\right) \operatorname{sinc}\left(\Omega\left(t - k\frac{\pi}{\Omega}\right)\right), \tag{1.1}$$

where $\operatorname{sinc}(t) = \sin(t)/t$, and Ω is the bandwidth of $f(t)$, i.e.,

$$f(t) = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} F(\omega) e^{i\omega t} d\omega. \tag{1.2}$$

Here $F(\omega)$ is the square integrable function on $(-\Omega, \Omega)$. In the sequel we shall denote the class of functions satisfying (1.2) by $\mathrm{BL}(\Omega)$.

* Corresponding author. Tel.: +1 204 474 8881; fax: +1 204 261 4639.
E-mail address: pawlak@ee.umanitoba.ca (M. Pawlak).

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In practical applications of the sampling formula in (1.1) one has to truncate the series in (1.1). Moreover, the samples $f(k\frac{\pi}{\Omega})$ are often observed in the presence of noise, i.e., we have noisy observations $y_k = f\left(k\frac{\pi}{\Omega}\right) + z_k$, where z_k is a zero mean noise process with a finite variance. This would lead to the following naive reconstruction scheme

$$f_n(t) = \sum_{|k|\leq n} y_k \operatorname{sinc}\left(\Omega\left(t - k\frac{\pi}{\Omega}\right)\right), \tag{1.3}$$

where $2n+1$ is the number of observations taken into account. The fundamental question, which arises is whether $f_n(t)$ can be a consistent estimate of $f(t)$, i.e., whether $\varrho(f_n, f) \to 0$ as $n \to \infty$, in a certain probabilistic sense, for some distance measure ϱ.

Since $f(t)$ is assumed to be square integrable, then the natural measure between $f_n(t)$ and $f(t)$ is the mean integrated square error

$$\mathrm{MISE}(f_n) = \mathbf{E}\int_{-\infty}^{\infty}(f_n(t) - f(t))^2 \mathrm{d}t. \tag{1.4}$$

This represents the energy of the error signal $f_n(t) - f(t)$.

It was shown in [5] that

$$\mathrm{MISE}(f_n) = \frac{\sigma^2\pi(2n+1)}{\Omega} + \frac{\pi}{\Omega}\sum_{|k|>n} f^2\left(k\frac{\pi}{\Omega}\right). \tag{1.5}$$

This readily implies that $\mathrm{MISE}(f_n) \to \infty$ as $n \to \infty$. This unpleasant property of the estimate $f_n(t)$ is caused by the presence of the noise process in the observed data and the fact that $f_n(k\frac{\pi}{\Omega}) = y_k$, i.e., $f_n(t)$ interpolates the noisy observations. It is clear that one should avoid interpolation schemes in the presence of noise since they would retain random errors.

The aim of this paper is to propose a consistent estimate of $f(t)$ being a smooth correction of the naive algorithm in (1.3). This task is carried out by sampling a signal at irregularly spaced quasi-random points and by carefully selecting the number of terms in the sampling series. The conditions for consistency of our estimate are established.

Throughout the paper we assume that the data are generated from the following model

$$y_k = f(\tau_k) + z_k, \quad k = 0, \pm 1, \pm 2, \ldots \tag{1.6}$$

where τ_k are quasi-random points, and z_k is uncorrelated noise process with $\mathbf{E}z_k = 0$, $\mathrm{var}(z_k) = \sigma^2 < \infty$.

The problem of recovering a nonparametric function in the model (1.6) has been an active area of research in recent years, see, e.g., [6,7]. Nonparametric curve estimation techniques like kernel, spline, orthogonal series estimates have been extensively studied. Nevertheless, estimated functions have been assumed to be defined on a finite interval. In a number of signal processing and communication systems problems, the assumption that $f(t)$ is defined on a finite interval has a little physical significance. In fact, the function $f(t)$ has to be interpreted as a signal defined everywhere in time and, moreover, the signal has finite energy and bounded frequency content. These requirements are met by the class $BL(\Omega)$ considered in our paper. Moreover, it has been commonly assumed that the sampling scheme in (1.6) is linear, i.e., the sampling points are specified as $\tau_k = k\Delta$, for some sampling period Δ. In this paper, we propose a nonlinear sampling scheme based on the theory of quasi-random sequences. The use of quasi-random sequences as the sampling scheme has several salient features that are listed below.

(1) If a quasi random-sequence is generated in the way as it is described in the next section, then adding new sampling points does not require changing positions of the sampling points that were placed earlier. The advantage of this property can be fully appreciated when one samples spatial objects like images. Note also that in the case of the regular sampling scheme with $\tau_k = k\Delta$ we need to rearrange all the previous samples.

(2) When a smoothing method is used to reconstruct a signal then the accuracy of the method is inevitably dependent on the quadrature error of approximating integrals by the corresponding finite sums. It is known that the quadrature error caused by quasi-random points is considerably smaller than that caused by the classical uniformly distributed sequences. This fact is extensively utilized in this paper.

(3) Samples taken at quasi-random points can be transmitted in a secure way, since they are useless unless an irrational number, which we use for designing quasi-random points, is known.

2. Sampling with quasi-random points

The notion of quasi-random sequences has been originally established in the theory of numerical integration (see [8,9]). A sequence of real numbers $\{x_j, j = 1, 2, \ldots\}$ is said to be a quasi-random sequence in $[0, 1]$ if for every function $b(x)$ that is continuous on $[0, 1]$ we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} b(x_j) = \int_0^1 b(x)\mathrm{d}x. \tag{2.1}$$