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The method of lower and upper solutions for 2*n*th-order multi-point boundary value problems

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ABSTRACT

In this paper, we consider the following 2nth-order multi-point boundary value problems

$$(-1)^n u^{(2n)}(t) = f(t, u(t), -u''(t), \ldots, (-1)^i u^{(2i)}(t), \ldots, (-1)^{n-1} u^{(2n-2)}(t)), \quad 0 < t < 1,$$

with the $2(n-m)$-point boundary conditions

$$u^{(2j)}(0) = u^{(2j)}(1) = 0, \quad j = 0, 1, 2, \ldots, m,\ m \le n-2,$$
$$u^{(2j)}(0) - a_j u^{(2j)}(\xi_j^{(1)}) = 0,$$
$$u^{(2j)}(1) - b_j u^{(2j)}(\xi_j^{(2)}) = 0, \quad j = m+1, \ldots, n-1,$$

where $a_j \ge 0$, $b_j \ge 0$, $0 < \xi_j^{(1)} < \xi_j^{(2)} \le 1$, $j = m+1, \ldots, n-1$. The existence of iterative solutions is obtained by using the lower and upper solution method for the above $2(n-m)$-point boundary value problems.

1. Introduction

In this paper, we are interested in the following 2nth-order multi-point boundary value problems

$$(-1)^n u^{(2n)}(t) = f(t, u(t), -u''(t), \ldots, (-1)^i u^{(2i)}(t), \ldots, (-1)^{n-1} u^{(2n-2)}(t)), \quad 0 < t < 1, \tag{1.1}$$

with the $2(n-m)$-point boundary conditions

$$\begin{cases} u^{(2j)}(0) = u^{(2j)}(1) = 0, \quad j = 0, 1, 2, \ldots, m,\ m \le n-2, \\ u^{(2j)}(0) - a_j u^{(2j)}(\xi_j^{(1)}) = 0, \\ u^{(2j)}(1) - b_j u^{(2j)}(\xi_j^{(2)}) = 0, \quad j = m+1, \ldots, n-1, \end{cases} \tag{1.2}$$

where $n > 1$ is an integer and f, a_j, b_j satisfy the following hypotheses:

(H_1) $1 > a_j \ge 0$, $1 > b_j \ge 0$ and $\rho_j = a_j \xi_j^{(1)}(1-b_j) + (1-a_j)(1-b_j\xi_j^{(2)}) > 0$, $j = m+1, \ldots, n-1$, $m \le n-2$;
(H_2) $f \in C([0,1] \times R^n, R)$, where $R = (-\infty, +\infty)$.

In the case of $n = 1$ and $n = 2$, the existence and multiplicity of solutions for the boundary value problems have been widely studied in [1–8] and references therein. Most of these results are based upon the Leray–Schauder continuation method, topological degree, the fixed point theorem on a cone, or the lower and upper solution method. Especially, in paper [6], the authors developed the monotone method for the following fourth-order boundary value problem

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$$u^{(4)}(x) = f(x, u(x), u''(x)), \quad x \in (0, 1),$$
$$u(0) = u(1) = u''(0) = u''(1) = 0.$$

Since then, some works have been done by using this method, see [4,6,15]. When n is an arbitrary natural number, boundary value problems have attracted considerable attention [9–13]. In paper [12], the authors considered the singular two-point boundary value problems of $2n$th-order ordinary differential equations

$$(-1)^n x^{(2n)}(t) = f(t, x(t), -x''(t), \ldots, (-1)^i x^{(2i)}(t), \ldots, (-1)^{n-1} x^{(2n-2)}(t)), \quad 0 < t < 1,$$
$$x^{(2i)}(0) = x^{(2i)}(1) = 0, \quad i = 0, 1, 2, \ldots, n-2,$$
$$ax^{(2n-2)}(0) - bx^{(2n-1)}(0) = 0, \qquad cx^{(2n-2)}(1) + dx^{(2n-1)}(1) = 0,$$

where $n > 1$. They gave a necessary and sufficient condition for the existence of $C^{2n-2}[0, 1]$ as well as $C^{2n-1}[0, 1]$ positive solutions.

To the best of our knowledge, there are few works that refer to iterative solutions for $2n$th-order multi-point differential equation boundary value problems. Therefore, motivated by [4,6,15], in this paper, we show the existence of iterative solutions for the problem (1.1)–(1.2) between a lower solution and an upper solution.

The remainder of the paper is organized as follows. In Section 2, we shall present some lemmas in order to prove our main results. Section 3 presents and proves our main results. In Section 4, we present an example to illustrate our results.

2. Preliminaries and lemmas

In order to prove our main results, we need the following lemmas and definitions.

Lemma 2.1 *([1]). If $\rho = a\xi(1-b) + (1-a)(1-b\eta) \neq 0$, $h(t) \in C[0, 1]$ and $h(t) \geq 0$, then the boundary value problem*

$$\begin{cases} u''(t) + h(t) = 0, \\ u(0) - au(\xi) = A, \qquad u(1) - bu(\eta) = B \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \frac{A - bA\eta + aB\xi}{\rho} + \frac{Ab + B - A - aB}{\rho} t + \int_0^1 G(t, s)h(s)\mathrm{d}s,$$

and $u(t) \geq 0$, where

$$G(t,s) = \begin{cases} s \in [0, \xi]: \begin{cases} \dfrac{s}{\rho}[(1-b\eta) + (b-1)t], & s \leq t; \\ \dfrac{t}{\rho}[(1-b\eta) + (b-1)s] + \dfrac{(\rho - 1 + b\eta)(s-t)}{\rho}, & t \leq s; \end{cases} \\ s \in [\xi, \eta]: \begin{cases} \dfrac{1}{\rho}[(1-b\eta) + (b-1)t](a\xi - as + s), & s \leq t; \\ \dfrac{1}{\rho}[(1-b\eta) + (b-1)s](a\xi - at + t), & t \leq s; \end{cases} \\ s \in [\eta, 1]: \begin{cases} \dfrac{1-s}{\rho}[t - at + a\xi] + (s-t), & s \leq t; \\ \dfrac{1-s}{\rho}[a\xi - at + t], & t \leq s. \end{cases} \end{cases} \tag{2.2}$$

Lemma 2.2. *Under the assumptions of* (H_1) *and* (H_2)*, if $u(t) \in C^{2n}[0, 1]$, satisfies*

$$(-1)^j u^{(2j)}(0) \geq 0, \qquad (-1)^j u^{(2j)}(1) \geq 0, \quad j = 0, 1, 2, \ldots, m,\ m \leq n-2,$$
$$(-1)^j [u^{(2j)}(0) - a_j u^{(2j)}(\xi_j^{(1)})] \geq 0,$$
$$(-1)^j [u^{(2j)}(1) - b_j u^{(2j)}(\xi_j^{(2)})] \geq 0, \quad j = m+1, \ldots, n-1,$$
$$(-1)^n u^{(2n)}(t) \geq 0, \quad t \in (0, 1).$$

Then $(-1)^j u^{(2j)}(t) \geq 0$, for $t \in [0, 1]$, $j = 0, 1, 2, \ldots, n-1$.

Proof. Let

$$(-1)^j u^{(2j)}(0) = x_0^{(j)}, \qquad (-1)^j u^{(2j)}(1) = x_1^{(j)}, \quad j = 0, 1, 2, \ldots, m, m \leq n-2,$$
$$(-1)^j \left[u^{(2j)}(0) - a_j u^{(2j)}(\xi_j^{(1)})\right] = x_2^{(j)},$$
$$(-1)^j \left[u^{(2j)}(1) - b_j u^{(2j)}(\xi_j^{(2)})\right] = x_3^{(j)}, \quad j = m+1, \ldots, n-1,$$
$$(-1)^n u^{(2n)}(t) = h(t), \quad t \in (0, 1),$$