Nonlinear Analysis 71 (2009) 4696–4704

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Symmetric ground state solutions of m-coupled nonlinear Schrödinger equations

Hichem Hajaiej

King Saud University, Department of Mathematics, College of Sciences, P.O. Box 2455, 11451 Riyadh, Saudi Arabia

ARTICLE INFO

Article history:
Received 19 July 2008
Accepted 13 March 2009

Keywords:
Schrödinger equations
Ground state
Cauchy problem

ABSTRACT

We prove the existence of radial and radially decreasing ground states of an m-coupled nonlinear Schrödinger equation with a general nonlinearity.

1. Introduction

The following is a Cauchy problem of an m-coupled nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t \Phi_1 + \Delta\Phi_1 + g_1\left(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2\right)\Phi_1 = 0, \\ \vdots \\ i\partial_t \Phi_m + \Delta\Phi_m + g_m\left(|x|, |\Phi_1|^2, \dots, |\Phi_m|^2\right)\Phi_m = 0, \\ \Phi_i(0,x) = \Phi_i^0(x) \quad \text{for } 1 \le i \le m. \end{cases} \tag{1.1}$$

For $1 \le i \le m : \Phi_i^0 : \mathbb{R}^N \to \mathbb{C}$ and $g_i : \mathbb{R}_+^* \times \mathbb{R}_+^m \to \mathbb{R}$, $\Phi_i : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{C}$, has numerous applications in physical problems. It appears in the study of spatial solitons in nonlinear waveguides [1], the theory of Bose–Einstein condensates [2], interactions of m-wavepackets [3], optical pulse propagation in birefringent fibers [4,5], wavelength division multiplexed optical systems. Physically, the solution Φ_i is the ith component of the beam in a Kerr-like photorefractive media [6]. In the most relevant cases, it is possible to write (1.1) in a vectorial form as follows:

$$\begin{cases} i\dfrac{\partial\Phi}{\partial t} = E'(\Phi) \\ \Phi(0,x) = \Phi^0 = (\Phi_1^0, \dots, \Phi_m^0) \end{cases} \tag{1.2}$$

where

$$E(\Phi) = \frac{1}{2}\|\nabla\Phi\|_2^2 - \int G\left(|x|, \Phi_1, \dots, \Phi_m\right)\, dx. \tag{1.3}$$

E-mail address: hichem.hajaiej@gmail.com.

0362-546X/$ – see front matter
doi:10.1016/j.na.2009.03.035

$G:(0,\infty)\times\mathbb{R}^m\to\mathbb{R}$ satisfies the following system:

$$\begin{cases}\dfrac{\partial G}{\partial u_1}=g_1\left(|x|,u_1^2,\ldots,u_m^2\right)u_1,\\ \vdots\\ \dfrac{\partial G}{\partial u_m}=g_m\left(|x|,u_1^2,\ldots,u_m^2\right)u_m.\end{cases}\tag{1.4}$$

When $m=1$, G can be easily given by the explicit expression: $G(r,s)=\frac{1}{2}\int_0^{s^2}g(r,t)\,\mathrm{d}t$.
In the general case:

$$\begin{aligned}G(r,u_1,\ldots,u_m)&=\frac{1}{2}\int_0^{u_1^2}g_1(r,t,u_2^2,\ldots,u_m^2)\,\mathrm{d}t+K_1(u_2,\ldots,u_m)\\&=\frac{1}{2}\int_0^{u_i^2}g_i(r,u_1^2,\ldots,t_i,\ldots,u_m^2)\,\mathrm{d}t_i+K_i(u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_m)\\&=\ldots\\&=\frac{1}{2}\int_0^{u_m^2}g_m(r,u_1^2,\ldots,t)\,\mathrm{d}t+K_m(u_1,\ldots,u_{m-1}).\end{aligned}\tag{1.5}$$

A soliton or standing wave of (1.1) is a solution of the form: $\Phi(t,x)=(\Phi_1(t,x),\ldots,\Phi_m(t,x))$, where for $1\le j\le m$: $\Phi_j(t,x)=u_j(x)\mathrm{e}^{-\mathrm{i}\lambda_j t}$, λ_j are real numbers. Therefore $\mathcal{U}=(u_1,\ldots,u_m)$ is a solution of the following $m\times m$ elliptic eigenvalue problem:

$$\begin{cases}\Delta u_1+\lambda_1u_1+g_1\left(|x|,u_1^2,\ldots,u_m^2\right)u_1=0,\\ \vdots\\ \Delta u_m+\lambda_mu_m+g_m\left(|x|,u_1^2,\ldots,u_m^2\right)u_m=0.\end{cases}\tag{1.6}$$

Among all the standing waves, let us mention the ground states which correspond to the least energy solutions $\mathcal{U}$ of (1.6), defined by:

$$E(\mathcal{U})=\frac{1}{2}\sum_{i=1}^m|\nabla u_i|_2^2-\int_{\mathbb{R}^N}G\left(|x|,u_1(x),\ldots,u_m(x)\right)\,\mathrm{d}x\tag{1.7}$$

under constraints

$$S_c=\left\{\mathcal{U}=(u_1,\ldots,u_m)\in\mathrm{H}^1(\mathbb{R}^N)\times\cdots\mathrm{H}^1(\mathbb{R}^N):\int_{\mathbb{R}^N}u_i^2=c_i\right\}\tag{1.8}$$

where $c_i>0$ are m prescribed numbers.
Ground states are solutions of the minimization problem:

$$\text{For given } c_i>0,\quad M_c=\inf_{\mathcal{U}\in S_c}E(\mathcal{U}).\tag{1.9}$$

Profiles of stable electromagnetic waves traveling along a medium are given by (1.9). Note that in (1.7), $|x|$ is the position relative to the optical axis, G is related to the index of refraction of the medium. In the most relevant cases, G has jumps at interfaces between layers of different media (core and claddings). Therefore, G is not continuous with respect to the first variable in many practical cases.

The existence of ground states has been investigated by many authors following different methods. In [7–17] by numerical arguments; in [18–23], the mathematical analysis using the variational approach has been pursued to prove the existence of ground states. These works addressed the special case $m=2$ and

$$\begin{cases}g_1(|x|,u_1^2,u_2^2)=\left(|u_1|^{2p-2}+\beta|u_1|^{p-2}|u_2|^p\right),\\ g_2(|x|,u_1^2,u_2^2)=\left(|u_2|^{2p-2}+\beta|u_2|^{p-2}|u_1|^p\right).\end{cases}\tag{1.10}$$

This is a very interesting case where we can easily determine G, indeed using (1.5) it is obvious that $G(r,s_1,s_2)=\frac{1}{2p}u_1^{2p}+\frac{\beta}{p}u_1^pu_2^p+K_1(u_2)=\frac{1}{2p}u_2^{2p}+\frac{\beta}{p}u_1^pu_2^p+K_2(u_1)$. A straightforward computation implies: $G(r,s_1,s_2)=\frac{1}{2p}s_1^{2p}+\frac{1}{2p}s_2^{2p}+\frac{\beta}{p}s_1^ps_2^p$.

In [18,22], not only the existence of ground states has been established, for (1.1) with g_i given by (1.10), but also the orbital stability has been discussed. Of course, we are interested in the orbital stability of ground states of (1.1) with general nonlinearities. However, an inescapable step consists in the establishment of suitable assumptions of g_i under which (1.1) admits a unique solution. This is a very challenging open question under investigation.