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Periodic bouncing solutions for attractive singular second-order equations

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ABSTRACT

In this paper, by using an approximation approach and phase-plane analysis for the successor map of the impact oscillator, we obtain an existence result for periodic bouncing solutions for a second-order forced equation, $x'' + g(x) = p(t)$ (where g has an attractive singularity), which improves on the previous result in the conditions for both g and p.

1. Introduction

In this paper, we will be concerned with the existence of periodic bouncing solutions for impact oscillators of the form

$$\begin{cases} x'' + g(x) = p(t), \quad x(t) > 0; \\ x(t) \geq 0; \\ x'(t_0+) = -x'(t_0-) \quad \text{for every } t_0 \text{ such that } x(t_0) = 0, \end{cases} \tag{1.1}$$

where p is a continuous 2π-periodic function, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying

$$\lim_{x\to 0^+} g(x) = +\infty, \qquad \lim_{x\to +\infty} g(x) = 0 \tag{1.2}$$

and the weak force condition

$$\int_0^1 g(s)\mathrm{d}s < +\infty. \tag{1.3}$$

Throughout this paper, we use the notation $\mathbb{R}^+ = (0, +\infty)$ and $\mathbb{R}_+ = [0, +\infty)$.

In [1], Lazer and Solimini initiated the study of the singular equation

$$x'' + \frac{1}{x^\alpha} = p(t)$$

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where $\alpha > 0$ and p is as above. By using upper and lower solutions, they proved that $\overline{p} := \frac{1}{2\pi}\int_0^{2\pi} p(t)\mathrm{d}t > 0$ is a necessary and sufficient condition for the existence of periodic solutions. This result was generalized in [2] to a wider class of equations, including the attractive singular equation $x'' + g(x) = p(t)$, where g satisfies condition (1.2).

From the classical result in [1,2], it is known that when $\overline{p} \leq 0$, classical periodic solutions of the above attractive singular equation do not exist. However the case of the equation with an obstacle (impact oscillator) has a different property. In fact, there are infinitely many so called bouncing periodic solutions for an attractive singular second-order impact oscillator (1.1).

Let us first recall the concept of the bouncing solution as follows [3].

Definition 1.1. A continuous function $x : \mathbb{R} \to \mathbb{R}_+$ is called a bouncing solution of Eq. (1.1) if there exists a double-infinity sequence $\{t_i\}_{i\in\mathbb{Z}}$ such that for all $i \in \mathbb{Z}$ the following conditions hold:

(1) $x(t_i) = 0$;
(2) $x'(t_i+) = -x'(t_i-)$;
(3) $x \in C^2((t_i, t_{i+1}), \mathbb{R}^+)$ is a classical solution of $x'' + g(x) = p(t)$.

Recently in [3], Qian and Torres studied the existence of bouncing periodic solutions of the second-order impact oscillator (1.1) with the attractive singularity under the weak force condition, i.e., $g(x)$ satisfies (1.2) and (1.3). For when $\overline{p} < 0$, they proved that if $g(x)$ is strictly decreasing when x is near zero, there exist infinitely many bouncing periodic solutions of (1.1). This result was obtained by using phase-plane analysis of the successor map for the impact oscillator and a generalized version of the Poincaré–Birkhoff theorem.

Comparing the results in [1,2], the condition "$g(x)$ is strictly decreasing when x is near zero" seems to be a technical hypothesis. In [3], this condition is used only to prove the uniqueness of the solution for the initial value problem (see Lemma 2.3 in [3]). Indeed, for the proof of the main result in this paper it is sufficient to provide the existence of a solution to the Cauchy problem associated with the equation in (1.1), and this can be done through an approximation argument and the Arzela–Ascoli theorem. There is also an open question in [3] as regards whether there exist infinitely many periodic bouncing solutions for (1.1) under the weak condition $\overline{p} = 0$ (see "Further remark" in [3]).

In this paper, we will remove the monotone hypothesis of g and also answer the open question above in the affirmative.

The approach that we used here is a little different from that in [3]. Motivated by the arguments in [4–6], we first investigate the existence of periodic bouncing solutions for the auxiliary equation $x'' + g_k(x) = p_k(t)$, where $g_k(x)$ and $p_k(t)$ are auxiliary functions tending to $g(x)$ and $p(t)$, respectively. Note that functions $g_k(x)$ are not necessary singular when x is at zero, so we can choose the g_k such that the solution for the initial value problem of the auxiliary equation is unique (that is why the strict monotonicity of g becomes unnecessary in this paper). Then we give a uniform estimate for these periodic bouncing solutions. Hence a subsequence of these solutions tends to a periodic bouncing solution of (1.1).

Using this approximation approach, we obtain the following result.

Theorem 1.1. *Assume that $\overline{p} \leq 0$, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a locally Lipschitz-continuous function satisfying (1.2) and (1.3). Then, for any $m, n \in \mathbb{N}$, there exists at least one $2m\pi$-periodic solution of (1.1) with n bounces in each period.*

The paper is organized as follows. In Section 2 we introduce some auxiliary equations for (1.1) and then prove that every solution of these equations with positive initial velocity is of bouncing type, and define the successor maps for these equations. Section 3 is devoted to the study of the existence of periodic bouncing solutions for auxiliary equations via a generalized version of the Poincaré–Birkhoff theorem. The main result is proved in Section 4 by using some analysis for limiting properties for periodic bouncing solutions obtained in Section 3.

2. Auxiliary equations and their successor maps

Since $g(x)$ satisfies the attractive condition (1.2), we have a sequence of numbers $\{x_k\}$ such that $x_k \to 0^+$ as $k \to \infty$ and $g(x_k) = k$, $g(x) \geq k$ for $0 < x < x_k$. Let $g_k(x) = g(x)$ for $x \geq x_k$, $g_k(x) = k$, for $0 < x < x_k$. Choose 2π-periodic continuous functions $p_k(t)$, $k = 1, 2, \ldots$, such that we have the mean values $\overline{p_k} < 0$ and $\|p_k - p\|_\infty \to 0$ as $k \to \infty$.

We will investigate the auxiliary impact problem of (1.1) in the following form:

$$\begin{cases} x'' + g_k(x) = p_k(t), \quad x(t) > 0; \\ x(t) \geq 0; \\ x'(t_0+) = -x'(t_0-) \quad \text{for every } t_0 \text{ such that } x(t_0) = 0. \end{cases} \tag{2.1}$$

Note that $g_k(x)$, $k = 1, 2, \ldots$, are locally Lipschitz-continuous functions without singularities and $p_k(t)$, $k = 1, 2, \ldots$, continuous functions, so we can denote by $x(t; \tau, v)$ the unique solution of the impact problem (2.1) with initial conditions $x(\tau; \tau, v) = 0$, $x'(\tau; \tau, v) = v > 0$, where $\tau \in \mathbb{R}$ and $v \in \mathbb{R}^+$. The following lemma shows that every solution above is of bouncing type, i.e., there is $\hat{\tau}$ in the maximal interval I_x of $x(\tau; \tau, v)$ such that

$$x(\tau; \tau, v) = x(\hat{\tau}; \tau, v) = 0, \qquad x'(\hat{\tau}; \tau, v) < 0 \quad \text{and} \quad x(s; \tau, v) > 0 \quad \text{for } s \in (\tau, \hat{\tau}).$$

Lemma 2.1. *Assume $\overline{p_k} < 0$, and $g_k : \mathbb{R}^+ \to \mathbb{R}^+$ is a locally Lipschitz-continuous function satisfying (1.3) and $\lim_{x\to+\infty} g_k(x) = 0$. Then for $k \gg 1$, the solution $x(t; \tau, v)$ of Eq. (2.1) with the initial condition $x(\tau; \tau, v) = 0$; $x'(\tau; \tau, v) = v > 0$ is defined*