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A Picone identity for half-linear elliptic equations and its applications to oscillation theory[☆]

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ABSTRACT

A Picone identity is established for a class of half-linear elliptic operators, and Sturmian comparison and oscillation results are derived on the basis of the Picone identity. Wirtinger inequalities and Riccati inequalities are also obtained as corollaries of the Picone identity, and a new class of half-linear elliptic equations is introduced.

1. Introduction

Since the pioneering work of Sturm [1] in 1836, efforts have been made to derive Sturmian comparison theorems for differential equations of various type. In particular, Picone [2] established the so-called Picone identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u}{v}\big(a(t)u'v - A(t)v'u\big)\right) = \big(a(t) - A(t)\big)(u')^2 + \big(C(t) - c(t)\big)u^2 + A(t)\left[v\left(\frac{u}{v}\right)'\right]^2 + \frac{u}{v}\big(vq[u] - uQ[v]\big)$$

to obtain Sturmian comparison theorems for ordinary differential operators q, Q defined by

$$q[u] = \big(a(t)u'\big)' + c(t)u,$$

$$Q[v] = \big(A(t)v'\big)' + C(t)v.$$

Picone identities play an important role in the study of qualitative theory of differential equations, and hence much interest has been focused on establishing Picone identities. We refer to Kreith [3,4], Swanson [5] for Picone identities and Sturmian comparison theorems for linear elliptic equations, and to Allegretto [6], Allegretto and Huang [7,8], Bognár and Došlý [9], Dunninger [10], Kusano, Jaroš and Yoshida [11], Yoshida [12,13] for Picone identities, Sturmian comparison and/or oscillation theorems for half-linear elliptic equations. In particular, we mention the paper [10] by Dunninger which seems to be the first paper dealing with Sturmian comparison theorems for half-linear elliptic equations.

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We are concerned with the half-linear elliptic operators p and P defined by

$$p[u] := \sum_{k=1}^{m} \nabla \cdot \left(a_k(x) \left| \sqrt{a_k(x)} \nabla u \right|^{\alpha-1} \nabla u \right) + c(x)|u|^{\alpha-1}u, \tag{1.1}$$

$$P[v] := \sum_{k=1}^{m} \nabla \cdot \left(A_k(x) \left| \sqrt{A_k(x)} \nabla v \right|^{\alpha-1} \nabla v \right) + C(x)|v|^{\alpha-1}v, \tag{1.2}$$

where $\alpha > 0$ is a constant, the dot $\cdot$ denotes the scalar product and $\nabla = \left(\partial/\partial x_1, \ldots, \partial/\partial x_n\right)^{\mathrm{T}}$ (the superscript T denotes the transpose). In case $m = 1$ and $a_1(x)$ is the identity matrix I_n, the principal part of (1.1) reduces to the p-Laplacian $\nabla \cdot \left(|\nabla u|^{p-2} \nabla u\right)$ $(p = \alpha + 1)$.

We note that for a real, symmetric, positive semidefinite [*resp*. positive definite] matrix $a(x)$ there exists a unique symmetric positive semidefinite [*resp*. positive definite] matrix $\sqrt{a(x)}$ satisfying $\left(\sqrt{a(x)}\right)^2 = a(x)$.

The notation $|x|$ will be used for the Euclidean norm of $x \in \mathbb{R}^n$, and the operator norm $\|M(x)\|_2$ of an $n \times n$ matrix function $M(x)$ will be defined by

$$\|M(x)\|_2 = \sup\{|M(x)\xi|;\ \xi \in \mathbb{R}^n,\ |\xi| \le 1\}.$$

It is known that

$$\|M(x)\|_2 = \sqrt{\lambda_{\max}(M(x)^{\mathrm{T}} M(x))},$$

where $\lambda_{\max}(M(x)^{\mathrm{T}} M(x))$ denotes the largest eigenvalue of $M(x)^{\mathrm{T}} M(x)$.

The purpose of this paper is to establish a Picone identity for the half-linear elliptic operators p and P, and to derive Sturmian comparison and oscillation theorems for these operators. As applications of the Picone identity, Wirtinger inequalities and Riccati inequalities are obtained.

In Section 2 we establish a Picone identity to obtain Sturmian comparison theorems for p and P. In Section 3 specializations of Sturmian comparison results to linear elliptic operators and to half-linear elliptic operators are discussed, and a new class of half-linear elliptic equations which are not previously studied is given. In Section 4 we present Wirtinger inequalities and Section 5 is devoted to oscillation results for $P[v] = 0$ via Picone identity. Riccati inequalities are derived to investigate the oscillation character of $P[v] = 0$ in Section 6.

2. Picone identity and Sturmian comparison theorems

Let G be a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G. It is assumed that the matrices $a_k(x), A_k(x) \in C(\overline{G}; \mathbb{R}^{n\times n})$ $(k = 1, 2, \ldots, m)$ are symmetric and positive semidefinite in G, and that $c(x), C(x)$ are of class $C(\overline{G}; \mathbb{R})$.

The domain $\mathcal{D}_p(G)$ of p is defined to be the set of all functions u of class $C^1(\overline{G}; \mathbb{R})$ satisfying $a_k(x) \left| \sqrt{a_k(x)} \nabla u \right|^{\alpha-1} \nabla u \in C^1(G; \mathbb{R}^n) \cap C(\overline{G}; \mathbb{R}^n)$. The domain $\mathcal{D}_P(G)$ of P is defined similarly.

Theorem 2.1. *If $v \in \mathcal{D}_P(G)$ and $v \neq 0$ in G (that is, v has no zero in G), then we obtain the following identity for any $u \in C^1(G; \mathbb{R})$:*

$$\begin{aligned} -\sum_{k=1}^{m} \nabla \cdot \left(u\varphi(u) \frac{A_k(x) \left| \sqrt{A_k(x)} \nabla v \right|^{\alpha-1} \nabla v}{\varphi(v)} \right) &= -\sum_{k=1}^{m} \left| \sqrt{A_k(x)} \nabla u \right|^{\alpha+1} + C(x)|u|^{\alpha+1} \\ &+ \sum_{k=1}^{m} \left[\left| \sqrt{A_k(x)} \nabla u \right|^{\alpha+1} + \alpha \left| \sqrt{A_k(x)}\, \frac{u}{v} \nabla v \right|^{\alpha+1} - (\alpha+1) \left(\sqrt{A_k(x)} \nabla u \right) \cdot \Phi\left(\sqrt{A_k(x)}\, \frac{u}{v} \nabla v \right) \right] \\ &- \frac{u}{\varphi(v)} \big(\varphi(u) P[v]\big), \end{aligned} \tag{2.1}$$

where $\varphi(s) = |s|^{\alpha-1}s$ $(s \in \mathbb{R})$ and $\Phi(\xi) = |\xi|^{\alpha-1}\xi$ $(\xi \in \mathbb{R}^n)$.

Proof. It is easy to see that

$$\begin{aligned} -\nabla \cdot \left(u\varphi(u) \frac{A_k(x) \left| \sqrt{A_k(x)} \nabla v \right|^{\alpha-1} \nabla v}{\varphi(v)} \right) &= -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(u\varphi(u) \frac{(A_k)_{ij}(x) \left| \sqrt{A_k(x)} \nabla v \right|^{\alpha-1} \frac{\partial v}{\partial x_j}}{\varphi(v)} \right) \\ &= -\sum_{i,j=1}^{n} \left[\frac{\partial}{\partial x_i} \big(u\varphi(u)\big) \frac{(A_k)_{ij}(x) \left| \sqrt{A_k(x)} \nabla v \right|^{\alpha-1} \frac{\partial v}{\partial x_j}}{\varphi(v)} + u\varphi(u) \left(-\frac{\varphi'(v)}{\varphi(v)^2} \frac{\partial v}{\partial x_i} \right) (A_k)_{ij}(x) \left| \sqrt{A_k(x)} \nabla v \right|^{\alpha-1} \frac{\partial v}{\partial x_j} \right. \\ &\left. + u\varphi(u) \frac{\frac{\partial}{\partial x_i} \left((A_k)_{ij}(x) \left| \sqrt{A_k(x)} \nabla v \right|^{\alpha-1} \frac{\partial v}{\partial x_j} \right)}{\varphi(v)} \right], \end{aligned} \tag{2.2}$$