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The transmission problem for quasi-linear elliptic second order equations in a conical domain. I, II

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ABSTRACT

We investigate the behavior of weak solutions to the transmission problem for quasi-linear elliptic divergence second order equations in a neighborhood of the boundary conical point. In Part I we study the problem for quasi-linear equation with semi-linear principal part. In Part 2 we study the problem for general quasi-linear elliptic divergence second order equations.

PART I

1. Introduction

The transmission problems often appear in different fields of physics and technics. For instance, one of the important problems of the electrodynamics of solid media is the electromagnetic process research in ferromagnetic media with different dielectric constants. These problems appear as well as in solid mechanics if a body consists of composite materials.

In this work we obtain the estimates of weak solutions of the nonlinear elliptic transmission problem near conical boundary point. Namely, for weak solutions of this problem we establish the possible exponent in the local bound of the solution modulus. Earlier the quasi-linear transmission problem was investigated only in smooth domains (see works of Borsuk [1], Rivkind–Ural'tseva [2], Kutev–Lions [3]). Later other mathematicians are studied transmission problems in non-smooth domains in some particular linear cases (see the references cited in [4–6]). General *linear* interface problems in polygonal and polyhedral domains were considered in [4,5]. Regularity results in terms of weighted Sobolev–Kondratiev spaces are obtained in [6] for two- and three-dimensional transmission problems for the Laplace operator. D. Kapanadze and B.-W. Schulze studied boundary-contact problems with conical [7] singularities and edge [8] singularities at the interfaces for general linear any order elliptic equations (as well as systems). They constructed parametrix and showed regularity with asymptotics of solutions in weighted Sobolev–Kondratiev spaces. We knew only one paper, namely D. Knees' work [9], that concerns with the study of the regularity of weak solutions of special nonlinear transmission problem on polyhedral

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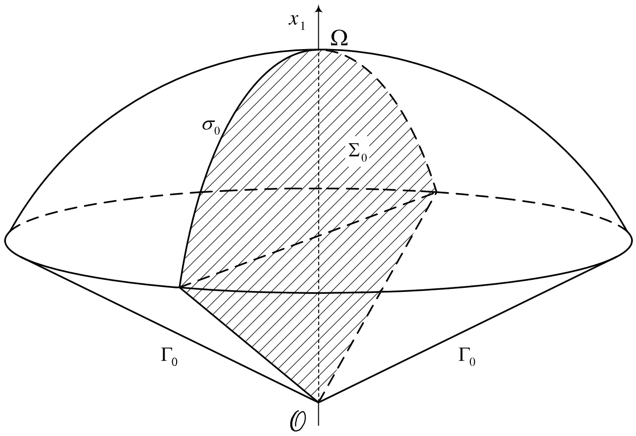

Fig. 1.

domains. A principal new feature of our work is the consideration of estimates of solutions for *general quasi-linear* second order equations in n-dimensional conic domains.

Let $G \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with boundary ∂G that is a smooth surface everywhere except at the origin $\mathcal{O} \in \partial G$ and near the point $\mathcal{O}$ it is a conical surface with vertex at $\mathcal{O}$. We assume that $G = G_+ \cup G_- \cup \Sigma_0$ is divided into two subdomains G_+ and G_- by a $\Sigma_0 = G \cap \{x_n = 0\}$, where $\mathcal{O} \in \overline{\Sigma_0}$. In Part I we consider the transmission problem for a quasi-linear equation with semi-linear principal part and establish the **best possible exponent** in the local bound of the solution modulus

$$\begin{cases} -\dfrac{d}{dx_i}\left(|u|^q a^{ij}(x)u_{x_j}\right) + b(x,u,\nabla u) = 0, \quad q \geq 0, x \in G \setminus \Sigma_0; \\ [u]_{\Sigma_0} = 0, \quad \mathcal{S}[u] \equiv \left[\dfrac{\partial u}{\partial \nu}\right]_{\Sigma_0} + \dfrac{1}{|x|}\sigma\left(\dfrac{x}{|x|}\right)u \cdot |u|^q = h(x,u), \quad x \in \Sigma_0; \\ \mathcal{B}[u] \equiv \dfrac{\partial u}{\partial \nu} + \dfrac{1}{|x|}\gamma\left(\dfrac{x}{|x|}\right)u \cdot |u|^q = g(x,u), \quad x \in \partial G \setminus \mathcal{O} \end{cases} \tag{WL}$$

(summation over repeated indices from 1 to n is understood); here (Fig. 1):

- $$u(x) = \begin{cases} u_+(x), & x \in G_+, \\ u_-(x), & x \in G_-; \end{cases} \qquad a^{ij}(x) = \begin{cases} a^{ij}_+(x), & x \in G_+, \\ a^{ij}_-(x), & x \in G_- \end{cases} \qquad \text{etc.};$$
- $[u]_{\Sigma_0} = u_+(x)|_{\Sigma_0} - u_-(x)|_{\Sigma_0}$, where $u_\pm(x)|_{\Sigma_0} = \lim_{G_\pm \ni y \to x \in \Sigma_0} u_\pm(y)$;
- $\frac{\partial}{\partial \nu} = a^{ij}(x)\cos(\overrightarrow{n}, x_i)\frac{\partial}{\partial x_j}$, where $\overrightarrow{n}$ denotes the unit outward with respect to G_+ (or G) normal to Σ_0 (respectively $\partial G \setminus \mathcal{O}$);
- $\left[\frac{\partial u}{\partial \nu}\right]_{\Sigma_0}$ denotes the saltus of the co-normal derivative of the function $u(x)$ on crossing Σ_0, i.e.
$$\left[\frac{\partial u}{\partial \nu}\right]_{\Sigma_0} = a^{ij}_+(x)\cos(\overrightarrow{n}, x_i)\left.\frac{\partial u}{\partial x_j}\right|_{\Sigma_0} - a^{ij}_-(x)\cos(\overrightarrow{n}, x_j)\left.\frac{\partial u}{\partial x_j}\right|_{\Sigma_0}.$$

We introduce the following notations:

- S^{n-1}: a unit sphere in $\mathbb{R}^n$ centered at $\mathcal{O}$;
- (r,ω), $\omega = (\omega_1, \omega_2, \ldots, \omega_{n-1})$: the spherical coordinates of $x \in \mathbb{R}^n$ with pole $\mathcal{O}$:
$$\begin{aligned} x_1 &= r\cos\omega_1, \\ x_2 &= r\cos\omega_2 \sin\omega_1, \\ &\vdots \\ x_{n-1} &= r\cos\omega_{n-1}\sin\omega_{n-2}\ldots\sin\omega_1, \\ x_n &= r\sin\omega_{n-1}\sin\omega_{n-2}\ldots\sin\omega_1. \end{aligned}$$
- $\mathcal{C}$: the convex rotational cone $\{x_1 > r\cos\frac{\omega_0}{2}\}$ with the vertex at $\mathcal{O}$;
- $\partial\mathcal{C}$: the lateral surface of $\mathcal{C} : \{x_1 = r\cos\frac{\omega_0}{2}\}$;