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# Duality and subdifferential for convex functions on complete *CAT*(0) metric spaces

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ABSTRACT

Thanks to the recent concept of quasilinearization of Berg and Nikolaev, we have introduced the notion of *duality* and *subdifferential* on complete *CAT*(0) (Hadamard) spaces. For a Hadamard space $X$, its dual is a metric space $X^*$ which strictly separates non-empty, disjoint, convex closed subsets of $X$, provided that one of them is compact. If $f : X \to (-\infty, +\infty]$ is a proper, lower semicontinuous, convex function, then the subdifferential $\partial f : X \rightrightarrows X^*$ is defined as a multivalued monotone operator such that, for any $y \in X$ there exists some $x \in X$ with $\overrightarrow{xy} \in \partial f(x)$. When $X$ is a Hilbert space, it is a classical fact that $\mathcal{R}(I + \partial f) = X$. Using a *Fenchel* conjugacy-like concept, we show that the approximate subdifferential $\partial_\epsilon f(x)$ is non-empty, for any $\epsilon > 0$ and any $x$ in efficient domain of $f$. Our results generalize duality and subdifferential of convex functions in Hilbert spaces.


## 1. Introduction

As Reich and Shafrir [1] have suggested, some kinds of *hyperbolic spaces* can be a suitable context for some notions in nonlinear analysis. Kirk has proposed in [2] that the *complete CAT*(0) *spaces* (usually called *Hadamard spaces*) can be successfully applied for this purpose, and has generalized some *fixed point theorems* to Hadamard spaces. A Hadamard space is a complete metric space $(X, d)$ which is satisfied in the following condition.

CAT(0)-INEQUALITY: For every two points $x_0, x_1 \in X$ and for every $0 < t < 1$ there exists some $x_t \in X$ such that

$$d^2(y, x_t) \le (1-t)d^2(y, x_0) + td^2(y, x_1) - t(1-t)d^2(x_0, x_1) \quad (y \in X). \tag{1}$$

For other equivalent definitions and basic properties, we refer the reader to standard texts such as [3–5]. In this paper, we generalize the notion of subdifferential for proper, semicontinuous, convex functions on Hadamard spaces. To this end, we introduce a *dual space* $X^*$ for a Hadamard space $X$, based on the recent work of Berg and Nikolaev [6]. It is well known that a normed linear space satisfies CAT(0)-inequality if and only if it is a pre-Hilbert space, hence it is not so unusual to have an *inner product-like* notion in Hadamard spaces. Berg and Nikolaev in [6,7] have introduced the concept of *quasilinearization* along these lines. Let us formally denote a pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a *vector*. Then quasilinearization is defined as a map $\langle , \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}d^2(a, d) + \frac{1}{2}d^2(b, c) - \frac{1}{2}d^2(a, c) - \frac{1}{2}d^2(b, d) \quad (a, b, c, d \in X). \tag{2}$$

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We say that $X$ satisfies the Cauchy–Schwarz inequality if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \le d(a,b)d(c,d) \quad (a,b,c,d \in X). \tag{3}$$

Berg and Nikolaev have then proved the following result [6, Corollary 3].

**Theorem 1.1.** *A geodesically connected metric space is CAT(0)-space if and only if it satisfies the Cauchy–Schwarz inequality.*

Also, we can formally add *compatible* vectors, more precisely $\overrightarrow{xy} + \overrightarrow{yz} = \overrightarrow{xz}$, for all $x, y, z \in X$. For more details see [6]. The *efficient domain* of a function $f : X \to (-\infty, +\infty]$ is $\mathcal{D}(f) = \{x \in X : f(x) < +\infty\}$ and the closure of a set $A \subseteq X$ is denoted by cl $A$.

## 2. Dual space

In order to define the conjugate space of a Hadamard space $X$, consider the map $\Theta : \mathbb{R} \times X \times X \to C(X)$ defined by

$$\Theta(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle \quad (t \in \mathbb{R}, a, b, x \in X) \tag{4}$$

where $C(X,\mathbb{R})$ is the space of all continuous real-valued functions on $X$. Then the Cauchy–Schwarz inequality implies that $\Theta(t,a,b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t,a,b)) = t\, d(a,b)$, for all $t \in \mathbb{R}$ and $a, b \in X$, where $L(\varphi) = \sup\{\frac{\varphi(x)-\varphi(y)}{d(x,y)}; x, y \in X, x \neq y\}$ is the Lipschitz semi-norm, for any function $\varphi : X \to \mathbb{R}$. Now, we introduce the pseudometric $D$ on $\mathbb{R} \times X \times X$ by

$$D((t,a,b),(s,c,d)) = L(\Theta(t,a,b) - \Theta(s,c,d)) \quad (t, s \in \mathbb{R}, a, b, c, d \in X). \tag{5}$$

**Lemma 2.1.** $D((t,a,b),(s,c,d)) = 0$ *if and only if* $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$, *for all* $x, y \in X$.

**Proof.** By (4) and (5) and definition of Lipschitz semi-norm, $D((t,a,b),(s,c,d)) = 0$ if and only if there exists a constant $k \in \mathbb{R}$ such that $t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{cx} \rangle + k$, for all $x \in X$. Therefore, for all $x, y \in X$,

$$t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{ay} \rangle - t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{cy} \rangle - s\langle \overrightarrow{cd}, \overrightarrow{cx} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle.$$

Conversely, if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$, for all $x, y \in X$, then

$$\Theta(t,a,b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{ax} \rangle = \Theta(s,c,d)(x) - s\langle \overrightarrow{cd}, \overrightarrow{ca} \rangle,$$

for all $x \in X$, which yields $D((t,a,b),(s,c,d)) = 0$. □

**Definition and Notation 2.2.** For a Hadamard space $(X,d)$, the pseudometric space $(\mathbb{R} \times X \times X, D)$ can be considered as a subspace of the pseudometric space $(\mathrm{Lip}(X,\mathbb{R}), L)$ of all real-valued Lipschitz functions. Also, $D$ defines an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of $(t,a,b)$ is

$$[t\overrightarrow{ab}] = \{s\overrightarrow{cd}; t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle (x, y \in X)\}.$$

The set $X^* := \{[t\overrightarrow{ab}]; (t,a,b) \in \mathbb{R} \times X \times X\}$ is a metric space with metric $D$, which is called the *dual metric space* of $(X,d)$.

Let us observe that if $X$ is a closed and convex subset of a Hilbert space $\mathcal{H}$ with non-empty interior, then $X^* = \mathcal{H}$. Without loss of generality let $B_\epsilon(0) \subset X$, for some $\epsilon > 0$, and define the map $\iota : \mathcal{H} \to X^*$ by $\iota(x) = \frac{2\|x\|}{\epsilon}\left[\overrightarrow{o\frac{\epsilon x}{2\|x\|}}\right]$, for $x \neq 0$, and $\iota(0) = \mathbf{0}$. We claim that $\iota$ is a surjective isomorphism. First observe that

$$\begin{aligned} D(\iota(x), \iota(y)) &= L\left(\Theta\left(\frac{2\|x\|}{\epsilon}, o, \frac{\epsilon x}{2\|x\|}\right) - \Theta\left(\frac{2\|y\|}{\epsilon}, o, \frac{\epsilon y}{2\|y\|}\right)\right) \\ &= \sup_{u \neq v}\left\{\frac{|(x\cdot u - y\cdot u) - (x\cdot v - y\cdot v)|}{\|u-v\|}\right\} \\ &= \sup_{u \neq v}\left\{\frac{|(x-y)\cdot(u-v)|}{\|u-v\|}\right\} = \|x-y\|, \end{aligned}$$

for each $x \neq 0$, $y \neq 0$, and

$$D(\iota(x), \iota(0)) = L\left(\Theta\left(\frac{2\|x\|}{\epsilon}, o, \frac{\epsilon x}{2\|x\|}\right)\right) = \sup_{u \neq v}\left\{\frac{|(x\cdot u - y\cdot u)|}{\|u-v\|}\right\} = \sup_{w \neq 0}\left\{\frac{|x\cdot w|}{\|w\|}\right\} = \|x\|,$$

for each $x \neq 0$.