Nonlinear Analysis 72 (2010) 2816–2820

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Uniqueness and asymptotic behavior of coexistence states for a non-cooperative model of nuclear reactors☆

Wenshu Zhou

Department of Mathematics, Dalian Nationalities University, 116600, China

ARTICLE INFO

Article history:
Received 27 July 2009
Accepted 10 November 2009

MSC:
35J55
35K45
35K57

Keywords:
Nuclear reactor
Coexistence state
Non-cooperative system
Asymptotic behavior
Uniqueness

ABSTRACT

In this paper, we analyze the asymptotic behavior of coexistence states for a non-cooperative model of nuclear reactors. In addition, we also present some remarks on the uniqueness of coexistence states in a high dimensional case. Our results complement the work of López-Gómez [J. López-Gómez, The steady states of a non-cooperative model of nuclear reactors, J. Differential Equations 246 (2009), 358–372].

1. Introduction

In the recent paper [1], López-Gómez studied the existence, uniqueness and asymptotic behavior of coexistence states for the system

$$\begin{cases} -\Delta u = au - buv & \text{in } \Omega, \\ -\Delta v = cu - duv - ev & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with boundary $\partial\Omega$ of class $C^{2+\nu}$ for some $\nu \in (0, 1)$, and the parameters a, b, c, d and e are positive. By a coexistence state (u, v) of problem (1) we mean that (u, v) is a positive solution of problem (1) such that $\partial u/\partial n, \partial v/\partial n < 0$ on $\partial\Omega$, where n stands for the outward unit normal on $\partial\Omega$. Also, for any measurable V in Ω, we denote by $\sigma[-\Delta + V]$ the principal eigenvalue of $-\Delta + V$ in Ω subject to homogeneous Dirichlet boundary conditions; $\sigma_1 := \sigma[-\Delta]$. Subsequently, the coexistence states of problem (1) are regarded as terms of the form (a, u, v), where $a > 0$ and $(u, v) \in [C_0^{2+\nu}(\overline{\Omega})]^2$ is a coexistence state of problem (1). According to Proposition 2.1 in [1], a positive solution of problem (1) must be a coexistence state.

It is well known that the parabolic problem corresponding to system (1) is a refinement of a former model proposed by Kastenberg and Chambré [2] for modeling a nuclear reactor, where u represents the density of fast neutrons and v represents the temperature. We refer to [3,2,4,5] for the more detailed background and [1] for some reviews. In [1], López-Gómez proved that problem (1) admits a coexistence state if and only if $\sigma_1 < a < \sigma_1 + bc/d$, and that problem (1) has at most a coexistence state in the case where $N = 1$. Moreover, as an important property of coexistence states as $a \uparrow \sigma_1 + bc/d$, he showed the

☆ Supported by the Department of Education of Liaoning Province (2009A152) and NNSFC (10901030).
E-mail address: wolfzws@163.com.

0362-546X/$ – see front matter
doi:10.1016/j.na.2009.11.025

following:

Theorem 1.1 (*Theorem 1.3 in [1]*). *Suppose*

$$bc/d > e + \sigma_1 \tag{2}$$

and let $\{a_n\}_{n\geqslant 1}$ *be a sequence such that*

$$\sigma_1 < a_n < \sigma_1 + bc/d, \qquad \lim_{n\to\infty} a_n = \sigma_1 + bc/d.$$

For each $n \geqslant 1$, *let* (a_n, u_n, v_n) *be a coexistence state of problem (1). Then*

$$\limsup_{n\to\infty} v_n(x) = c/d \tag{3}$$

and

$$\lim_{n\to\infty} u_n(x) = \infty \quad \text{uniformly on compact subsets of } \Omega. \tag{4}$$

According to Theorem 1.1, the density of fast neutrons blows up as a approximates the threshold $\sigma_1 + bc/d$ provided that (2) holds. By an inspection of the proof to Theorem 1.1, it is easy to see that the condition (2) is only used to show (4), and (3) still holds without the restriction. We point out that the restriction is essential for the method used to show (4) since it relies on the fact that the problem

$$-\Delta w = \lambda w - w^2 \quad \text{in } \Omega,\ w|_{\partial\Omega} = 0$$

has a unique positive solution if and only if $\lambda > \sigma_1$ (cf. [6]). In the present paper, we remove the restriction and show the same conclusion (4). In addition, we also prove a result stronger than (3). Unlike that of [1], our method is based on Lemma 2.2 in Section 2. Our theorem can be stated as follows:

Theorem 1.2. *Let* $\{a_n\}_{n\geqslant 1}$ *be a sequence such that*

$$\sigma_1 < a_n < \sigma_1 + bc/d, \qquad \lim_{n\to\infty} a_n = \sigma_1 + bc/d.$$

For each $n \geqslant 1$, *let* (a_n, u_n, v_n) *be a coexistence state of problem (1). Then, as* $n \to \infty$,

$$v_n(x) \to c/d \text{ strongly in } L^p(\Omega) \quad \text{for any } p \geqslant 1,$$

and

$$u_n(x) \to \infty \quad \text{uniformly on compact subsets of } \Omega.$$

Moreover, it holds that

$$\lim_{n\to\infty} \left(u_n/\|u_n\|_{C(\overline{\Omega})}\right) = \phi_1 \quad \text{uniformly on } \overline{\Omega},$$

where ϕ_1 is the eigenfunction associated with σ_1 with $\|\phi_1\|_{C(\overline{\Omega})} = 1$.

Finally, we give some remarks on the uniqueness of positive solutions of problem (1) in the case where $N \geqslant 2$ and $f = \frac{bc}{d} - a - e \geqslant 0$. Let (u, v) be a positive solution of problem (1). Inspired by [3], we set $w = \frac{b}{d}v - u$. Then it follows from the equations for u and v that (u, w) satisfies

$$\begin{cases} -\Delta u = au - du(w+u) & \text{in } \Omega, \\ -\Delta w = fu - ew & \text{in } \Omega, \\ u = w = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

In the case where $f > 0$, the existence, uniqueness and stability of positive solutions of problem (5) was discussed in [7]. For some related studies one can refer to [8,9] and references therein. For $f > 0$, since $u > 0$ on $\overline{\Omega}$ and $e > 0$, by the maximum principle, it follows from the second equation of (5) that $w > 0$ on $\overline{\Omega}$. By Theorem 4 in [7], if $af \leqslant 2e^2$, i.e. $a^2 + a(e - bc/d) + 2e^2 \geqslant 0$, then a positive solution of problem (5) is uniquely determined; moreover, the solution is asymptotically stable. In the case where $f = 0$, since $e > 0$, by the comparison theorem, it follows from the second equation of (5) that $w = 0$, i.e. $u = \frac{b}{d}v$. Substituting $v = \frac{d}{b}u$ into the equation for u, we see that u must satisfy

$$-\Delta u = au - du^2 \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega.$$

It is well known that the above problem admits a unique positive solution if $a > \sigma_1$ (cf. [6]). Consequently, by means of Proposition 2.1 in [1], we arrive at the following conclusion.

Theorem 1.3. *Suppose that* $N \geqslant 2$ *and* $bc/d - e \geqslant a > \sigma_1$. *Then:*

(1) *If* $a = bc/d - e$, *then problem (1) admits a unique coexistence state* (u, v) *such that* $u = \frac{b}{d}v$ *and* u *is the unique positive solution of problem*

$$-\Delta u = au - du^2 \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega.$$