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Reconstruction of two time independent coefficients in an inverse problem for a phase field system

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ABSTRACT

In this paper we present stability results concerning the inverse problem of determining two time independent coefficients for a phase field system in a bounded domain $\Omega \subset \mathbb{R}^n$ for the dimension $n \leq 3$ with a single observation on a subdomain $\omega \Subset \Omega$ and the Sobolev norm of certain partial derivatives of the solutions at a fixed positive time $\theta \in (0, T)$ over the whole spatial domain. The proof of these results relies on an appropriate Carleman estimate for the phase field system.

1. Introduction

Solving inverse problem is a task that arises often in many branches of applied mathematics where the values of some model parameters must be obtained from the internal or boundary measurements. These problems arise, for example, in most geophysical investigations searching for minerals, oil and gas, in nondestructive evaluation of materials and in many other applied areas. The main objective of this paper is to establish a stability result for determining two time independent coefficients for coupled parabolic equations with Dirichlet boundary data arising from phase transitions.

In the mathematical literature, free boundary problems arising from phase transitions have been studied for more than a century. Most of the work is concerned with the classical Stefan problem [1] which incorporates the physics of latent heat and heat diffusion in a homogeneous medium. Recall that several authors have discussed phase field models, first introduced by Caginalp [2] and Fix [3], and improved the results from the thermodynamical point of view; see, for instance, the work by Penrose and Fife [4] for an exhaustive explanation of the underlying physics. And, in fact, these works provide an extension of the enthalpy method for the Stefan problem with the advantage of making it possible to describe some rather fine physical phenomena which can take place during fusion–solidification processes. The Stefan problem as well as other classical mathematical models of phase transitions are limiting cases of this system. Moreover, in recent years, study of several variants of the model has been carried out and interesting results have been obtained in the directions of existence and regularity of solutions as well as of their dependence on the physical parameters.

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In this work we consider the following linearized phase field model describing the phase transitions between two states, for example, solid or liquid, in pure material:

$$\left.\begin{array}{ll} u_t + l(x)v_t - \Delta u + a(x)v = f_1(x), & \text{in } Q = \Omega \times (0,T), \\ v_t - \Delta v + b(x,t)v + c(x)u = f_2(x), & \text{in } Q, \\ u(x,\theta) = u_\theta(x),\ v(x,\theta) = v_\theta(x), & \text{in } \Omega, \\ u(x,t) = h_1(x,t),\ v(x,t) = h_2(x,t), & \text{on } \Sigma = \partial\Omega \times (0,T), \end{array}\right\} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ for the dimension $n \leq 3$ and with boundary $\partial\Omega$ of class C^2. The coefficient $l \in L^\infty(\Omega)$ is the latent heat, $b \in C^1(\bar{Q})$ and for some fixed $\theta \in (0,T)$, the semi-initial values $u_\theta, v_\theta : \Omega \to \mathbb{R}$ are sufficiently regular (for instance, $(u_\theta, v_\theta) \in (H^2(\Omega))^2$). The nonzero smooth Dirichlet boundary data $h_1, h_2 : \Sigma \to \mathbb{R}$ are kept fixed and $(f_1, f_2) \in (L^2(\Omega))^2$ are given functions. The solution u denotes the temperature distribution of a material which occupies the region Ω and can be in either of the two phases, solid or liquid (if the melting temperature is taken to be zero), and the smooth function v is called as the "phase field function" and it is scaled so that v near $+1$ is associated with the liquid phase and v near -1 is associated with the solid phase. The interface is defined implicitly as the set of points for which v vanishes (see [2,5]). The unknown coefficients $a(x)$ and $c(x)$ are assumed to be sufficiently smooth and will be kept independent of time t.

The goal of this paper is to obtain the Lipschitz stability estimate in determining the coefficients $a(x)$ and $c(x)$ by means of an internal measurement of only one observation on a bounded domain for the dimension $n \leq 3$. The key ingredient of these stability results is an L^2-weighted inequality of Carleman type for solutions of the phase field systems which will be explained briefly later. Null controllability of the phase field system by two control forces has been studied by Barbu [6] while the same model with one control force has been studied by Ammar Khodja et al. [7]. Moreover, the recent paper by Xu and Liu [8] studies the exact controllability of the trajectories for semilinear heat equations with superlinear nonlinearity in a bounded domain of $\mathbb{R}^N$ for any $N \geq 1$. We follow the method used in [6] with different transformations related to inverse problems to obtain a Carleman estimate with two observations and we deduce a new Carleman estimate with one observation by using certain energy type estimates.

As far as the stability estimate of an inverse problem for parabolic equations via Carleman estimates is concerned, there exist a vast number of publications. Referencing all these works is beyond the scope of this paper. So let us mention briefly some of the interesting results. The theory of the Carleman inequality is one of the fastest developing areas of partial differential equations (PDEs); in particular, after the pioneering work of Carleman in 1939, the theory of inequalities of Carleman type was rapidly developed and now many general results are available for partial differential equations. The method of Carleman estimates was introduced in the field of inverse problems for the first time by Bukhgeim and Klibanov [9–11]. The paper by Klibanov [12] presents a brief review of the applications of Carleman estimates to inverse problems for PDEs with respect to three fundamental issues, namely, uniqueness, stability and numerical methods. Interestingly, the book by Klibanov and Timonov [13] gives a rigorous treatment of obtaining the numerical solution of non-overdetermined coefficient inverse problems with the lateral data via Carleman estimates for both linear and nonlinear partial differential equations. In this direction, a globally convergent numerical method for multidimensional coefficient inverse problems for hyperbolic PDEs have been established by Beilina and Klibanov [14] (see also other cited references therein) by using certain Carleman weight functions in the numerical scheme.

Further, we note that Carleman estimation is an essential technique for establishing the unique continuation and also for solving the exact controllability of parabolic equations; see, for example, Imanuvilov [15], Fursikov and Imanuvilov [16], Sakthivel et al. [17,18]. But we restrict ourselves to the literature related to inverse problems only. Imanuvilov and Yamamoto [19] determined a source term for general parabolic equations where the source term satisfies $g_t(x,t) \leq Cg(x,\theta)$, for any $\theta \in (0,T)$. For the determination of f by means of a single measurement using Carleman estimates, we also refer the reader to Bukhgeim and Klibanov [10], Bukhgeim [20], Klibanov [11,21], Isakov [22], where conormal derivatives are taken as overdetermining data. Yamamoto and Zou [23] investigated the simultaneous reconstruction of the initial temperature and heat radiative coefficient in a heat conductive system. Baudouin and Mercado [24] discussed the inverse problem for the Schrödinger equations with discontinuous coefficients and Poisson [25] studied the heat equation with discontinuous diffusion coefficients. The inverse problem for certain parabolic equations with memory has been studied by Baranibalan et al. [26]. Moreover, Benabdallah et al. [27] gives uniqueness and stability results for both the diffusion coefficient and the initial condition for the heat equation from a measurement of the solution on an arbitrary part of the boundary and at some arbitrary positive time. Recently, the inverse coefficient problem for a nonlinear parabolic variational inequality with an unknown leading coefficient in the equation for the gradient of the solution has been discussed by Hasanov and Liu [28].

Apart from the literature mentioned above for inverse problems for parabolic equations, recently, global Carleman estimates have been applied to one-measurement (or two-measurement) inverse parabolic systems of variable and constant coefficients. For example, Cristofol et al. [29] discuss the simultaneous reconstruction of one coefficient and initial conditions for the reaction–diffusion system from the measurement of one solution over $(t_0, T) \times \Omega$ and some measurement at fixed time $\hat{T} \in (t_0, T)$, and Benabdallah et al. [30] studies the inverse problems of determining all (or some of) the coefficients from the observations on an arbitrary subdomain over a time interval of only one component and data for two components at a fixed positive time θ over the whole spatial domain. The paper by Baranibalan et al. [31] studies the inverse problem of determining a diffusion coefficient for a phase field system of two equations from one observation and Sakthivel et al. [32] successfully apply a similar method for determining two diffusion coefficients of a Lotka–Volterra competitive diffusion system of three equations with two observations.