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Generalized 1D Jörgens theorem

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ABSTRACT

We study the Cauchy problem for a 1D nonlinear wave equation on $\mathbb{R}$. The nonlinearity can depend on the unknown function and its first order spatial derivative. Using the fixed point theorem we prove the existence of a classical solution. Moreover, the existence of periodic and almost periodic solutions are shown.

1. Introduction

In this paper we will study a nonlinear equation of the form

$$u_{tt} - u_{xx} = f(t, x, u, u_x) \quad \text{on } (0, T) \times \mathbb{R}. \tag{1}$$

We shall deal with the Cauchy problem for the above equation on $(0, T) \times \mathbb{R}$. It means to find a solution u of Eq. (1) such that

$$u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x) \quad \text{on } \mathbb{R}. \tag{2}$$

The main goal of the present paper is showing the existence and uniqueness of the classical solution to problem (1), (2). Moreover, we shall show the existence of periodic and almost periodic solutions to the periodic and almost periodic data respectively. Our result is a generalization of the Jörgens theorem (see [1]). In fact, Jörgens has worked with the 3D wave equation, where the nonlinearity has depended only on u.

The proof of our theorem relies on the application of the d'Alembert formula with Duhamel's principle. What is more, we use fixed point theorem in a carefully chosen Banach space.

2. Main result

We shall show that problem (1) poses a unique local in time classical solution. We distinguish three cases of initial conditions: arbitrary, periodic and almost periodic data. Before we present the main result we give the class of admissible nonlinearities.

Definition 1. We shall say that the map $f : \mathbb{R}^4 \to \mathbb{R}$ of C^2-class is admissible if for every compact set $K \subset \mathbb{R}^2$ the condition

$$\sup_{x \in \mathbb{R}^2, y \in K} \left(|f(x, y)| + |Df(x, y)| + |D^2 f(x, y)| \right) < \infty$$

holds.

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Moreover, let us fix $k \in \mathbb{N}$. We shall denote by $C^k(\mathbb{R})$ the space of k-times differentiable functions such that the norm

$$\|w\|_{C^k(\mathbb{R})} := \sum_{n=0}^{k} \sup_{x\in\mathbb{R}} \left|w^{(n)}(x)\right|$$

is finite.

2.1. Arbitrary data

In this subsection we shall work with the problem (1), (2) on the real line. It means:

$$\begin{aligned} &u_{tt} - u_{xx} = f(t, x, u, u_x) \quad \text{on } (0, T) \times \mathbb{R}, \\ &u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x) \quad \text{on } \mathbb{R}. \end{aligned} \tag{3}$$

Now, we can present the main result of this subsection.

Theorem 1. *Let us assume that f is admissible and $u_0 \in C^2(\mathbb{R})$, $u_1 \in C^1(\mathbb{R})$. Then there exist $T > 0$ and a unique solution $u \in C^2([0, T) \times \mathbb{R})$ of (3). Moreover, if $T < \infty$, then*

$$\max\left(\sup_{t\in[0,T),x\in\mathbb{R}} |u(t, x)|, \sup_{t\in[0,T),x\in\mathbb{R}} |u_x(t, x)|\right) = \infty$$

and T is minimal with this property.

Let us mention that the special case of the theorem has been shown in the paper [2].

Proof. First of all, for each $T \in (0, 1]$ we introduce the following space:

$$\begin{aligned} \tilde{C}^2([0, T) \times \mathbb{R}) = \Big\{ &u \in C^1([0, T] \times \mathbb{R}) : u_{xx}, u_{tx}, u_{xt} \in C^0([0, T) \times \mathbb{R}), \\ &\|u\|_{C^1([0,T]\times\mathbb{R})}, \|u_{xx}\|_{C^0([0,T)\times\mathbb{R})}, \|u_{tx}\|_{C^0([0,T)\times\mathbb{R})}, \|u_{xt}\|_{C^0([0,T)\times\mathbb{R})} < \infty \Big\}, \end{aligned}$$

endowed with the norm:

$$\|u\|_{\tilde{C}^2([0,T)\times\mathbb{R})} = \|u\|_{C^1([0,T]\times\mathbb{R})} + \|u_{xx}\|_{C^0([0,T)\times\mathbb{R})} + \|u_{tx}\|_{C^0([0,T)\times\mathbb{R})} + \|u_{xt}\|_{C^0([0,T)\times\mathbb{R})}.$$

Next, we define the function $\tilde{u}_0$ as follows

$$\tilde{u}_0(t, x) = \frac{1}{2}\left(u_0(x + t) + u_0(x - t)\right) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\,dy.$$

Subsequently, we define the set

$$X_T = \left\{ u \in \tilde{C}^2([0, T) \times \mathbb{R}) : u(0, x) = u_0(x), u_t(0, x) = u_1(x), \left\|u - \tilde{u}_0\right\|_{C^1([0,T]\times\mathbb{R})} \le 1, \left\|u_{xx} - \tilde{u}_{0\,xx}\right\|_{C^0([0,T)\times\mathbb{R})} \le 1 \right\}.$$

Next, we define the map $\mathcal{F}$:

$$\begin{aligned} &\mathcal{F} : X_T \longrightarrow C^2([0, T) \times \mathbb{R}), \\ &\mathcal{F}(v) = u, \end{aligned}$$

as follows: for $v \in X_T$ we define u as unique solution to the linear problem:

$$\begin{aligned} &u_{tt} - u_{xx} = f(t, x, v, v_x) \quad \text{on } (0, T) \times \mathbb{R}, \\ &u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x) \quad \text{on } \mathbb{R}. \end{aligned} \tag{4}$$

The map $\mathcal{F}$ is well defined. Indeed, since $v \in \tilde{C}^2([0, T) \times \mathbb{R})$ we conclude that the function $f(t, x, v, v_x)$ is C^1 and the problem (4) poses unique C^2 solution. Moreover, we can apply d'Alembert formula with the Duhamel's principle (see [3]) and the map $\mathcal{F}$ can be written as follows:

$$\mathcal{F}(v)(t, x) = \tilde{u}_0(t, x) + \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} f(t, y, v, v_y)\,dy\,d\tau.$$

Subsequently, let us introduce the notation:

$$x_a^b = x + b - a.$$

Now, we will show that there exists $T > 0$ such that $\mathcal{F} : X_T \to X_T$. We have to estimate the quantity:

$$\begin{aligned} \left\|\mathcal{F}(u) - \tilde{u}_0\right\|_{C^1([0,T]\times\mathbb{R})} = {} &\left\|\mathcal{F}(u) - \tilde{u}_0\right\|_{C^0([0,T]\times\mathbb{R})} + \left\|\partial_t\left(\mathcal{F}(u) - \tilde{u}_0\right)\right\|_{C^0([0,T]\times\mathbb{R})} \\ &+ \left\|\partial_x\left(\mathcal{F}(u) - \tilde{u}_0\right)\right\|_{C^0([0,T]\times\mathbb{R})} = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3. \end{aligned}$$