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# Elliptic regularization for the semi-linear telegraph system

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ABSTRACT

The solution of the semi-linear telegraph system is compared with the solution of an elliptic regularization, to which one associates two-point boundary conditions. An asymptotic approximation for the solution of the elliptic regularization is constructed. The method employed here is the boundary function method due to Vishik and Lyusternik. The problem is singularly perturbed of elliptic–hyperbolic type. To conduct this analysis, high regularity with respect to $t$ for the solutions of both problems is required. Finally, the order of this approximation is found in different spaces of functions.



## 1. Introduction

In the present paper we are concerned with the nonlinear hyperbolic system (S), the boundary conditions (BC) and the initial conditions (IC) given below:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial v}{\partial x} + F(x, u(t,x)) = f(t,x), & (t,x) \in Q = (0,T) \times (0,1) \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial u}{\partial x} + G(x, v(t,x)) = g(t,x), & (t,x) \in Q = (0,T) \times (0,1), \end{cases} \tag{S}$$

$$u(t,0) = u(t,1) = 0, \quad 0 < t < T \tag{BC}$$

$$u(0,x) = a_1(x), \qquad v(0,x) = a_2(x), \quad 0 < x < 1. \tag{IC}$$

Denote this initial-boundary value problem by $(T_0)$. Note that (S) is the well-known telegraph system which appears in electrical transmission line phenomena and in the integrated circuits theory. Problem $(T_0)$ also models some phenomena from mechanics and hydraulics: the unsteady fluid flow through pipes and the fluid flow through a tree-structured system of transmission pipelines [1].

The minimal hypotheses we assume are the following:

(C1) $F, G : [0,1] \times \mathbb{R} \to \mathbb{R}$, $F(\cdot, \xi), G(\cdot, \xi) \in L^2(0,1)$, $(\forall)\xi \in \mathbb{R}$;

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(C2) functions $\xi \mapsto F(x,\xi)$, $\xi \mapsto G(x,\xi)$ are continuous and nondecreasing for a.a. $x \in (0,1)$;
(C3) $a_1 \in H_0^1(0,1)$, $a_2 \in H^1(0,1)$;
(C4) $(f,g) \in W^{1,1}(0,T;L^2(0,1)^2)$.

Consider the real Hilbert space $H = L^2(0,1)^2$ with the scalar product

$$\left(\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right)_H = \int_0^1 [p_1(x)q_1(x) + p_2(x)q_2(x)]\mathrm{d}x$$

and the corresponding norm

$$\left\| \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \right\|_H = \left[ |p_1|^2_{L^2(0,1)} + |p_2|^2_{L^2(0,1)} \right]^{1/2}.$$

As a consequence of Proposition 5.1.1 and Theorem 5.1.1 from [2], we have the following results:

**Proposition 1.1.** *Under conditions* (C1), (C2), *the operator* $A : D(A) \subset H \to H$ *defined by*

$$A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v' + F(\cdot,u) \\ u' + G(\cdot,v) \end{pmatrix}, \qquad D(A) = H_0^1(0,1) \times H^1(0,1)$$

*is maximal monotone in* $H = L^2(0,1)^2$.

**Theorem 1.2.** *If conditions* (C1)–(C4) *hold, then problem* $(T_0)$ *has a unique strong solution*

$$(u,v)^T \in W^{1,\infty}(0,T;L^2(0,1)^2) \quad \text{with } u_x, v_x \in L^\infty(0,T;L^2(0,1)).$$

Here the upper symbol $T$ from $\alpha^T$ means the transposed of $\alpha$.

We compare the solution $(u,v)^T$ of $(T_0)$ with the solution $(u^\varepsilon, v^\varepsilon)^T$ of its elliptic regularization $(T_\varepsilon)$, consisting of the system $(S_\varepsilon)$, the boundary conditions $(BC_\varepsilon)$ and the initial conditions $(IC_\varepsilon)$ given below:

$$\begin{cases} \varepsilon \dfrac{\partial^2 u^\varepsilon}{\partial t^2} - \dfrac{\partial u^\varepsilon}{\partial t} = \dfrac{\partial v^\varepsilon}{\partial x} + F(x, u^\varepsilon(t,x)) - f(t,x), & (t,x) \in Q \\ \varepsilon \dfrac{\partial^2 v^\varepsilon}{\partial t^2} - \dfrac{\partial v^\varepsilon}{\partial t} = \dfrac{\partial u^\varepsilon}{\partial x} + G(x, v^\varepsilon(t,x)) - g(t,x), & (t,x) \in Q, \end{cases} \tag{$S_\varepsilon$}$$

$$u^\varepsilon(t,0) = u^\varepsilon(t,1) = 0, \quad 0 < t < T \tag{$BC_\varepsilon$}$$

$$\begin{cases} u^\varepsilon(0,x) = a_1(x), & v^\varepsilon(0,x) = a_2(x), \quad 0 < x < 1 \\ u^\varepsilon(T,x) = b_1(x), & v^\varepsilon(T,x) = b_2(x), \quad 0 < x < 1. \end{cases} \tag{$IC_\varepsilon$}$$

Denoting $w^\varepsilon = \begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix}$, $h = \begin{pmatrix} f \\ g \end{pmatrix}$, $w_0 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $w_T = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, the perturbed problem $(T_\varepsilon)$ can be written as the following two-point boundary value problem:

$$\begin{cases} \varepsilon \dfrac{\partial^2 w^\varepsilon}{\partial t^2} - \dfrac{\partial w^\varepsilon}{\partial t} = Aw^\varepsilon - h, \quad 0 < t < T \\ w^\varepsilon(0) = w_0, \qquad w^\varepsilon(T) = w_T. \end{cases} \tag{1.1}$$

Therefore, the existence and the regularity of the solutions for problem $(T_\varepsilon)$ can be analyzed with the aid of the theory of nonlinear second order differential equations of monotone type in Hilbert spaces. To this end, one imposes the following condition on the boundary values $b_1, b_2$:

(C5) $b_1 \in H_0^1(0,1)$, $b_2 \in H^1(0,1)$.

Applying the existence theory from [3], hypotheses (C1)–(C5) ensure that problem $(T_\varepsilon)$ has a unique solution $w^\varepsilon = (u^\varepsilon, v^\varepsilon)^T \in W^{2,2}(0,T;H)$. Problem (1.1) without the first derivative was studied in [4] in Banach spaces with some specific properties. Asymptotic behavior of solutions for evolution equations of type (1.1) was discussed in [5–8].

Using the boundary layer function method of Vishik and Lyusternik [9–12], one constructs an asymptotic approximation for the solution $(u^\varepsilon, v^\varepsilon)^T$ to the perturbed problem $(T_\varepsilon)$ with the aid of the solution $(u,v)^T$ to $(T_0)$. This asymptotic approximation is available in the entire set $Q$. The problem in discussion is singularly perturbed of elliptic–hyperbolic type. We will also prove that the solution of $(T_0)$ is approximated (as $\varepsilon \to 0$) by the solution of the perturbed problem $(T_\varepsilon)$, which is a smoother function. This justifies the advantage of using the elliptic regularization. Moreover, we find the order of approximation for the difference $u^\varepsilon - u$ and $v^\varepsilon - v$ with respect to $\varepsilon$, both in $L^2(Q)$ and in $C([0,\delta];H)$ for arbitrary, but fixed $\delta \in (0,T)$.