Nonlinear Analysis 72 (2010) 3112–3118

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Further regularity results for almost cubic nonlinear Schrödinger equation☆

S. Demirbaş [a,*], A. Eden [a], E. Kuz [b]

[a] Department of Mathematics, Boğaziçi University, Bebek 34342, Istanbul, Turkey

[b] Department of Mathematics, University of Maryland, College Park, MD 20742-4015, USA

ARTICLE INFO

Article history:
Received 13 September 2009
Accepted 20 November 2009

MSC:
35G25
35B53
35Q55

Keywords:
H^s well-posedness
Cubic NLS
Bourgain spaces
Besov spaces

ABSTRACT

We present three results related with the regularity of solutions of the almost cubic NLS. In the first one, following Ozawa's idea, we establish mass and energy conservation for the solutions without regularizing the initial datum. Our second result is the H^s well-posedness for the Cauchy problem for $0 < s < 1$. Finally, we show that the same solutions are also in some Bourgain spaces for possibly a smaller time interval. In all of our results, the non-local nonlinear term in the equation is shown to act like a cubic nonlinearity on the appropriate Sobolev and Besov spaces.

1. Introduction

The study of the almost cubic nonlinear Schrödinger (ACNLS) equations was initiated in the work of Eden and Kuz [1]. The study of this class of equations was motivated by a problem related with a Generalized Davey–Stewartson (GDS) system that is posed in Babaoğlu, Eden and Erbay [2]. In the purely elliptic case of the GDS system, the analysis done in that work failed to classify the focusing and defocusing cases of the equations. ACNLS equation was introduced with the hope of resolving this problem. The question was finally settled in Eden, Gürel and Kuz [3]. These results extend the classical ones for the usual cubic nonlinear Schrödinger (NLS) equations in two space dimensions. The similarities between the ACNLS and the usual NLS equations are yet to be further explored. In this note we try to take some further steps in that direction. Our aim in this note is threefold, first we show that the idea of Ozawa [4] for obtaining the conservation of mass and energy for the solutions without regularizing the data and the equations still applies to the ACNLS. The idea of Ozawa basically is to look at the solution as the fixed point of the appropriate map and perform the regularization within the map. As will be the theme in all the other parts of our paper the non-local nonlinearity that appears in the ACNLS equation has almost the same behaviour on function spaces as the local cubic nonlinearity and this might be putting our terminology almost cubic in jeopardy. Some of these estimates were already noted in [1] with (3.1), (3.6) and (3.7) and will be used in the second section. In the third section, we derive estimates for the non-local nonlinearity acting on Besov spaces. These estimates are enough to implement the idea of Cazenave–Weissler [5] for the local H^s well-posedness, Theorem 3.1. In the last section, we consider further regularity properties of the solutions with initial value in H^s in the spirit of Bourgain [6]. We prove in

☆ This work is supported in part by TUBITAK (Turkish Science and Technology Council) and BUVAK (Bogazici University Foundation).

* Corresponding author.
E-mail addresses: seckin.demirbas@boun.edu.tr (S. Demirbaş), eden@boun.edu.tr (A. Eden), elifkuz@gmail.com (E. Kuz).

0362-546X/$ – see front matter Crown Copyright © 2009 Published by Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2009.11.050

Theorem 4.3 that the solutions really reside in the Bourgain dispersive spaces. These results can be considered as a prelude for possible scattering results for H^s solutions for s close to 1 as it is done in Bourgain in [6] (see also Guo and Cui [7]).

The notation to be used throughout the text is as follows: For $k \in \mathbb{Z}$, $p \geq 1$, $W^{k,p} = W^{k,p}(\mathbb{R}^2; \mathbb{C})$. H^k is used instead of $H^k(\mathbb{R}^2; \mathbb{C})$ when $p = 2$. $(\cdot, \cdot)_{L^2}$ denotes the usual inner product in L^2 and $(\cdot, \cdot)_{4/3,4}$, $(\cdot, \cdot)_{W^{1,4/3}, W^{-1,4}}$ and $(\cdot, \cdot)_{H^{-1}, H^1}$ stand for the duality pairing in L^4, $W^{1,4/3}$ and H^1 respectively. We use $\| \cdot \|_p$ in order to denote the $L^p(\mathbb{R}^2; \mathbb{C})$-norm. $H^s(=H^s(\mathbb{R}^2))$, $s \in \mathbb{R}$ stands for the Banach space of elements $u \in \boldsymbol{S}'$ such that $(1 + |\boldsymbol{\xi}|^2)^{\frac{s}{2}} \hat{u} \in L^2$ and it is equipped with the norm $\|u\|_{H^s} = \|(1 + |\boldsymbol{\xi}|^2)^{\frac{s}{2}} \hat{u}\|_2$ where $\boldsymbol{S}$ is the Schwarz space with functions defined on $\mathbb{R}^2$ and $\hat{u}$ stands for the Fourier transform of u. We will also be using the Besov space $B^s_{p,q}(=B^s_{p,q}(\mathbb{R}^2))$, $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, being the Banach space of elements $u \in \boldsymbol{S}'$ with the norm

$$\|u\|_{B^s_{p,q}} = \|\mathcal{F}^{-1}(\eta \hat{u})\|_{L^p} + \begin{cases} \left(\displaystyle\sum_{j=1}^{\infty} (2^{sj} \|\mathcal{F}^{-1}(\psi_j \hat{u})\|_{L^p})^q \right)^{1/q} & \text{if } q < \infty, \\ \displaystyle\sup_{j \geq 1} 2^{js} \|\mathcal{F}^{-1}(\psi_j \hat{u})\|_{L^p} & \text{if } q = \infty, \end{cases}$$

where η is a radial function satisfying $\eta \in C_c^\infty(\mathbb{R}^2)$ such that $\eta(\boldsymbol{\xi}) = 1$ for $|\boldsymbol{\xi}| \leq 1$ and $\eta(\boldsymbol{\xi}) = 0$ for $|\boldsymbol{\xi}| \geq 2$, $\psi_j(\boldsymbol{\xi}) = \psi(2^{-j}\boldsymbol{\xi})$ for $\psi(\boldsymbol{\xi}) = \eta(\boldsymbol{\xi}) - \eta(2\boldsymbol{\xi})$ and $\mathcal{F}^{-1}f$ stands for the inverse Fourier transform of f when it makes sense. The notations $\dot{H}^s$ and $\dot{B}^s_{p,q}$ are used for the homogeneous versions of the corresponding spaces. For $p \geq 1$, $L^p(I; B)$, $I \subset \mathbb{R}$ denotes the Banach space of (classes of) measurable functions $u : I \to B$ such that $\|u\|_B \in L^p(I)$ where B is any Banach space. J^s and D^s are for the Bessel and Riesz potentials respectively and they are defined as $J^s u = \mathcal{F}^{-1}(1 + |\boldsymbol{\xi}|^2)^{\frac{s}{2}} \hat{u}$, $D^s u = \mathcal{F}^{-1}|\boldsymbol{\xi}|^s \hat{u}$. $(S(t))_{t \in \mathbb{R}}$ represents the solution semigroup for the linear equation $\mathrm{i}u_t + \delta u_{xx} + u_{yy} = 0$ (S depends on $\delta = \pm 1$) and finally $a \vee b$, $a \wedge b$ denote $\max\{a, b\}$ and $\min\{a, b\}$ respectively (see [14] for further details on the function spaces).

2. Conservation laws without regularization

In [1] we considered the Cauchy problem for the following two dimensional nonlinear Schrödinger equation:

$$\mathrm{i}u_t + \delta u_{xx} + u_{yy} = K(|u|^2)u, \quad \delta = \pm 1 \tag{2.1}$$

where $\widehat{K(f)}(\boldsymbol{\xi}) = \alpha(\boldsymbol{\xi})\hat{f}(\boldsymbol{\xi})$ for $f \in L^2$. Here, the symbol α satisfies:

(H1) α is real, even and homogeneous of degree 0,
(H2) $\alpha \in C^\infty(\mathbb{R}^2 \setminus \{(0, 0)\})$.

We called this equation an almost cubic nonlinear Schrödinger equation (ACNLS) and classified the cases $\delta = \pm 1$ as the elliptic and the hyperbolic cases. We have established local well-posedness of the corresponding Cauchy problem in L^2, H^1 and H^2 ([1, Theorems 3.5, 4.4, 7.6]). We have made use of this local theory to show conservation of mass and energy conservations. Mass and energy were given by

$$M(u) = \int_{\mathbb{R}^2} |u|^2 \mathrm{d}x\mathrm{d}y, \qquad E(u) = \int_{\mathbb{R}^2} \left[\delta|u_x|^2 + |u_y|^2 + \frac{1}{2} K(|u|^2)|u|^2 \right] \mathrm{d}x\mathrm{d}y, \tag{2.2}$$

respectively and mass is naturally defined for L^2 solutions, whereas it is possible to define energy for H^1 solutions. Conservation of these quantities was important in obtaining results on the global behaviour of the solutions (see e.g. [1, Corollary 4.5, Proposition 6.1]).

For the mass conservation, if we begin with H^1 solutions, considering H^{-1}–H^1 duality product of (2.1) with $2u$ gives $2\mathrm{i}(u_t, u)_{H^{-1}, H^1} = 2(\delta\|u_x\|_2^2 + \|u_y\|_2^2) + 2\int_{\mathbb{R}^2} K(|u|^2)|u|^2 \mathrm{d}x\mathrm{d}y$. Since the right hand side is real, we obtain the mass conservation on $[0, T^*)$. For the conservation of energy, multiplying (2.1) by $2\bar{u}_t$ and then taking the real parts gives $2\,\mathrm{Re}\bar{u}_t(\delta u_{xx} + u_{yy}) = K(|u|^2)(|u|^2)_t$, from which it follows that

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} (\delta|u_x|^2 + |u_y|^2) \mathrm{d}x\mathrm{d}y + \mathrm{Re} \int_{\mathbb{R}^2} \alpha(\boldsymbol{\xi}) \hat{f}(\boldsymbol{\xi}) \overline{\widehat{(f_t)}}(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \left[\int_{\mathbb{R}^2} (\delta|u_x|^2 + |u_y|^2) \mathrm{d}x\mathrm{d}y + \frac{1}{2} \int_{\mathbb{R}^2} K(|u|^2)|u|^2 \mathrm{d}x\mathrm{d}y \right], \end{aligned}$$

by using the Parseval identity and the fact that α is even. These formal computations are exact for H^2 solutions. Using continuous dependence results, one can approximate L^2 and H^1 solutions with H^1 and H^2 solutions respectively to obtain the necessary conservations. This is the path followed in the above mentioned paper. An alternative approach is due to Ozawa, [4]. Instead of approximating the solution with more regular solutions for which the conservation laws follow from formal computations, we can consider the integral equation corresponding to (2.1). The definition of K with the assumptions (H1) and (H2) allows us to deduce the facts that $\mathrm{Im}(K(|u|^2)|u|^2) = 0$ and for $G(u) \equiv \frac{1}{4}\int_{\mathbb{R}^2} K(|u|^2)|u|^2 \mathrm{d}x\mathrm{d}y$, $G \in C^1(L^4; \mathbb{R})$ with $G'(u)(v) = \mathrm{Re}\int_{\mathbb{R}^2} K(|u|^2)u\bar{v}\mathrm{d}x\mathrm{d}y$ for every $v \in L^4$.