Nonlinear Analysis 72 (2010) 3119–3123

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Uniformly continuous composition operators in the space of bounded φ-variation functions[☆]

J.A. Guerrero [a,*], H. Leiva [b], J. Matkowski [c], N. Merentes [d]

[a] Universidad Nacional Experimental del Táchira, Dpto. de Matemáticas y Física, San Cristóbal, Venezuela
[b] Universidad de los Andes, Escuela de Matemáticas, Mérida, Venezuela
[c] Institute of Mathematics, University of Zielona Góra, Zielona Góra, Poland
[d] Universidad Central de Venezuela, Escuela de Matemáticas, Caracas, Venezuela

ARTICLE INFO

Article history:
Received 10 September 2009
Accepted 20 November 2009

MSC:
47B33
26B30
26B40

Keywords:
φ-variation in the sense of Wiener
Uniformly continuous operator
Regularization
Composition operator
Jensen equation

ABSTRACT

We prove that, under some general assumptions, a generator of any uniformly continuous Nemytskii operator, mapping a subset of space of bounded variation functions in the sense of Wiener into another space of this type, must be an affine function. As a special case, we obtain an earlier result from Matkowski (in press) [4].

1. Introduction

Let X, Y be real normed spaces and C be a closed convex subset of X. For a fixed real interval I denote by X^I (or Y^I) the set of all functions $f : I \longrightarrow X$ (or $f : I \longrightarrow Y$). If $h : I \times C \longrightarrow Y$ is a given function, then the operator $H : X^I \longrightarrow Y^I$ defined by the formula

$$(Hf)(t) = h(t, f(t)), \quad t \in I \tag{1}$$

is called the Nemytskii composition operator generated by the function h.

Let $(BV_\varphi(I, X), \|\cdot\|_\varphi)$ be the Banach space of functions $f : I \to X$ which are of bounded φ-variation in the sense of Wiener, where the norm $\|\cdot\|_\varphi$ is defined with the aid of Luxemburg–Nakano–Orlicz seminorm [1–3].

Assume that H maps the set of functions $f \in BV_\varphi(I, X)$ such that $f(I) \subset C$ into $BV_\varphi(I, Y)$. In the present paper, we prove that, if H is uniformly continuous, then the left and right regularization of its generator h with respect for the first variable are affine functions in the second variable. This extends the main result of paper [4].

☆ This work was supported by the CDHT-ULA-project: C-1667-09-05-AA.

* Corresponding author. Tel.: +58 276 3961819.
E-mail addresses: jaguerrero4@gmail.com, jguerre@unet.edu.ve (J.A. Guerrero), hleiva@ula.ve (H. Leiva), J.Matkowski@wmie.uz.zgora.pl (J. Matkowski), nmer@ciens.ucv.ve (N. Merentes).

0362-546X/$ – see front matter
doi:10.1016/j.na.2009.11.051

2. Preliminaries

In this section we present some definitions and preliminary results related with bounded φ-variation functions in the sense of Wiener.

Let $\mathcal{F}$ be the set of all convex functions $\varphi : [0,\infty) \longrightarrow [0,\infty)$ such that: $\varphi(0^+) = 0$ and $\lim_{t\longrightarrow\infty} \varphi(t) = \infty$. Then we have that

Remark 2.1. If $\varphi \in \mathcal{F}$, then φ is continuous and strictly increasing. Indeed, the continuity of φ at each point $t > 0$ follows from its convexity and continuity at 0 from the assumption $\varphi(0) = \varphi(0^+) = 0$. Suppose that $\varphi(t_1) \geq \varphi(t_2)$ for some $0 < t_1 < t_2$. Then

$$\frac{\varphi(t_1)-\varphi(0)}{t_1-0} = \frac{\varphi(t_1)}{t_1} > \frac{\varphi(t_2)}{t_2} = \frac{\varphi(t_2)-\varphi(0)}{t_2-0},$$

contradicting the convexity of φ.

Definition 2.2. Let $\varphi \in \mathcal{F}$ and $(X, |\cdot|)$ be a real normed space. A function $f \in X^I$ is of bounded φ-variation in the sense of Wiener in I, if

$$v_\varphi(f) = v_\varphi(f, I) := \sup_\xi \sum_{i=1}^{m} \varphi\big(|f(t_i) - f(t_{i-1})|\big) < \infty, \tag{2}$$

where the supremum is taken over all increasing finite sequences $\xi = (t_i)_{i=0}^m$, $t_i \in I$, $m \in \mathbb{N}$.

For $\varphi(t) = t^p$ $(t \geq 0, p \geq 1)$, condition (2) coincides with the classical concept of variation in the sense of Jordan [5, Chapter 8] whenever $p = 1$, and in the sense of Wiener [6] if $p > 1$. The general Definition 2.2 was introduced by Young [7].

It is known that for all $a, b, c \in I$, $a \leq c \leq b$ we have $v_\varphi(f, [a, c]) \leq v_\varphi(f, [a, b])$ (that is, v_φ is increasing with respect to the interval) and $v_\varphi(f, [a, c]) + v_\varphi(f, [c, b]) \leq v_\varphi(f, [a, b])$.

We will denote by $V_\varphi(I, X)$ the set of all functions $f \in X^I$ with bounded φ-variation in Wiener sense. This is a symmetric and convex set; but it is not necessarily a linear space. In fact, Musielak and Orlicz proved the following statement: this class of functions is a vector space if, and only if, φ satisfies the δ_2 condition [8]. We denote by $BV_\varphi(I, X)$ the linear space of all functions $f \in X^I$ such that $v_\varphi(\lambda f) < \infty$ for some constant $\lambda > 0$.

In the linear space $BV_\varphi(I, X)$, the function $\|\cdot\|_\varphi$ defined by

$$\|f\|_\varphi := |f(a)| + p_\varphi(f), \quad f \in BV_\varphi(I, X),$$

where

$$p_\varphi(f) := p_\varphi(f, I) = \inf\Big\{\epsilon > 0 : v_\varphi\big(f/\epsilon\big) \leq 1\Big\}, \quad f \in BV_\varphi(I, X), \tag{3}$$

is a norm (see for instance [8]).

For $X = \mathbb{R}$, the linear normed space $(BV_\varphi(I, \mathbb{R}), \|\cdot\|_\varphi)$ was studied by Musielak and Orlicz [8], Ciemnoczołowski and Orlicz [9], and Maligranda and Orlicz [10]. In particular, it is shown in [10] that the space $(BV_\varphi(I, \mathbb{R}), \|\cdot\|_\varphi)$ is a Banach algebra. The functional $p_\varphi(\cdot)$ defined by (3) is called *the Luxemburg–Nakano–Orlicz seminorm* [1–3].

In what follows, the symbol $BV_\varphi(I, C)$ stands for the set of all functions $f \in BV_\varphi(I, X)$ such that $f : I \longrightarrow C$ and C is a subset of X.

Lemma 2.3 (*Chistyakov [11, Lemma 1]). For $f \in BV_\varphi(I, X)$, we have:*

(a) *if $t, t' \in I$, then $\|f(t) - f(t')\| \leq \varphi^{-1}(1)p_\varphi(f)$;*
(b) *if $p_\varphi(f) > 0$ then $v_\varphi\big(f/p_\varphi(f)\big) \leq 1$;*
(c) *for $\lambda > 0$,*
 (c1) *$p_\varphi(f) \leq \lambda$ if and only if $v_\varphi\big(f/\lambda\big) \leq 1$;*
 (c2) *if $v_\varphi\big(f/\lambda\big) = 1$ then $p_\varphi(f) = \lambda$.* ■

Property (a) in Lemma 2.3 implies that any function $f \in BV_\varphi(I, X)$ is bounded. Indeed, we have $\|f\| \leq \|f(a)\| + \|f(t) - f(a)\|$, whence

$$\|f\|_\infty \leq \|f(a)\| + \varphi^{-1}(1)p_\varphi(f) < \infty.$$

If $(X, |\cdot|)$ is a Banach space and $f \in BV_\varphi(I, X)$, then

$$f^-(t) := \lim_{s\uparrow t} f(s), \quad t \in I^-,$$

exists and is called the *left regularization* of f [12].

Let $BV_\varphi^-(I, X)$ denote the subset in $BV_\varphi(I, X)$ that consists of those functions that are left continuous on $I^- := I \setminus \{\inf I\}$.

Lemma 2.4 (*Chistyakov [11, Lemma 6]). If X is a Banach space and $f \in BV_\varphi(I, X)$, then $f^- \in BV_\varphi^-(I, X)$.* ■

Thus, if a function has a bounded φ-variation, then its left regularization is a left continuous function.