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S-asymptotically ω-periodic and asymptotically ω-periodic solutions to semi-linear Cauchy problems with non-dense domain

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ABSTRACT

In this work, we study the existence and uniqueness of S-asymptotically ω-periodic and asymptotically ω-periodic solutions to a first-order differential equation with linear part dominated by a Hille–Yosida operator with non-dense domain. Applications to partial differential equations, fractional integro-differential and neutral differential equations are given.

1. Introduction

The study of existence of almost periodic, asymptotically almost periodic, almost automorphic, asymptotically almost automorphic, pseudo-almost periodic and pseudo-almost automorphic solutions is one of the most attracting topics in the qualitative theory of differential equations due to its mathematical interest and to the applications in physics, mathematical biology and control theory, among other areas (see [1–12] and the references therein). However, most of these problems need to be studied in abstract spaces and operators are defined over non-dense domain. In this context the literature is very scarce (see [13–21]).

The literature concerning S-asymptotically ω-periodic functions with values in Banach spaces is very new. Recently four interesting articles were published. The first one, by Henríquez et al. [22], concerning to developing a theory of these types of functions in Banach spaces setting. In particular, the authors have established a relationship between S-asymptotically ω-periodic function and the class of asymptotically ω-periodic functions. In, [23, Lemma 2.1], it is established that a scalar S-asymptotically ω-periodic function is asymptotically ω-periodic. In the second article, by Nicola and Pierre [24], the authors provided two examples which show that the above assertion is false. The third article, by Henríquez et al. [25], concerning to the existence and qualitative properties of S-asymptotically ω-periodic mild solutions for some class of abstract neutral functional differential equations with infinite delay. The fourth work, by Cuevas and de Souza [26], is concerned with the existence of S-asymptotically ω-periodic mild solutions of fractional integro-differential equations (see also [27,28]).

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In this work, we study the existence and uniqueness of S-asymptotically ω-periodic and asymptotically ω-periodic solutions for a class of abstract differential equations described in the form

$$x'(t) = Ax(t) + f(t, x(t)), \quad t \geq 0, \tag{1.1}$$

$$x(0) = x_0, \tag{1.2}$$

where A is an unbounded linear operator, assumed to be Hille–Yosida (see Definition 2.5) of negative type, having the domain $D(A)$, not necessarily dense, on some Banach space X; and $f : [0, \infty) \times X_0 \to X$ is a continuous function, where $X_0 = \overline{D(A)}$. The regularity of solutions for (1.1) with $t \in \mathbb{R}$ in the space of pseudo-almost periodic (respectively, almost automorphic, pseudo-almost automorphic, compact almost automorphic) was considered in [18] (resp. [15,16]). The existence of S-asymptotically ω-periodic and asymptotically ω-periodic solutions for evolution equations with a linear part dominated by a Hille–Yosida of negative type operator constitutes until now an untreated topic in the literature. Many questions in connection with these kinds of equations remain unanswered. This fact is the main motivation of this work. Hille–Yosida operators turn out to be important for the study of abstract Cauchy problem in nonreflexive spaces (see Proposition 2.1). We remark that there are a great variety of semi-linear differential equations with a linear part dominated by a non-densely defined operator. Being non-dense arises, for example, from restrictions made on the space where the equation is considered (e.g., periodic continuous functions, Hölder continuous functions) and from boundary conditions (e.g., the set of continuous functions with null value on the boundary is non-dense in the space of continuous functions). The topological method that we have chosen to study existence of S-asymptotically ω-periodic solutions is the theory of fixed points, which has been a very powerful and important tool to the study of nonlinear phenomena. Specifically, we will use contraction mapping principle, Leray–Schauder alternative and Krasnoselkii's theorem. We remark that the latter two requires some compactness conditions; thus, to use these, some compactness assumptions must be imposed on the perturbation f in Eq. (1.1). We emphasize that the implementation of this approach is a priori not trivial.

To build intuition and throw some light on the power of our results and methods, we examine sufficient conditions for the existence and uniqueness of S-asymptotically ω-periodic and asymptotically ω-periodic solutions to a control system (see Proposition 3.2) and some partial differential equations (see Propositions 3.3 and 3.4). On the other hand, we use the machinery developed in Section 3 (see Theorem 3.3) to give a result on the existence of S-asymptotically ω-periodic solution of the semi-linear fractional integro-differential equation

$$v'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Bv(s)\mathrm{d}s + f(t, v(t)), \quad t \geq 0, \tag{1.3}$$

$$v(0) = u_0 \in X, \tag{1.4}$$

where $1 < \alpha < 2$, $B : D(B) \subset X \to X$ is a linear densely defined operator of sectorial type. In [26] the authors proved the existence and uniqueness of an S-asymptotically ω-periodic solution of the problem (1.3)–(1.4) using an operator theoretical approach and contraction mapping principle. However, those results are not sharp enough to include more sophisticated perturbations (see Theorem 3.6). We also discuss sufficient conditions for the existence and uniqueness of S-asymptotically ω-periodic solutions to neutral differential equations (see Section 3.3.4). The reader will perceive through this work that our abstract results and techniques suggest that it is possible to develop a nice connection with other applications.

We will now present a summary of this work. The second section provides the definitions and preliminaries results to be used in theorems stated and proved in this article. In particular, we review some of the standard properties of the S-asymptotically ω-periodic functions and we recall some basic results on extrapolation spaces, which are of fundamental importance in this work. The abstract extrapolation spaces have been used for various purposes, for example to study Volterra integro-differential equations and retarded differential equations (see [29,30]). The third section is divided into three parts. In the first one, Section 3.1, we obtain very general results on the existence of S-asymptotically ω-periodic mild solutions to the problem (1.1)–(1.2). The second part is concerned with the existence of asymptotically ω-periodic mild solutions to (1.1)–(1.2) (see Section 3.2). In the third part (Section 3.3), we give a few applications to control systems, partial differential equations and neutral differential equations. In particular, we offer an interesting improvement on the results established in [26] for semi-linear fractional integro-differential equations (see Section 3.3.3). Finally in Section 3.3.5 we consider linear systems with exponential dichotomy. This kind of dichotomy gives relevant information about S-asymptotically ω-periodicity of solutions for several perturbed quasi-linear systems.

2. Preliminaries

Let $(X, \|\cdot\|)$ be Banach space. In this work $C_b([0, \infty); X)$ denotes the space consisting of the continuous and bounded functions from $[0, \infty)$ into X, endowed with the norm of the uniform convergence which is denoted for $\|\cdot\|_\infty$. Let us recall the notion of S-asymptotically ω-periodic functions which will be used later on.

Definition 2.1 (*[22]*). A function $f \in C_b([0, \infty); X)$ is called S-asymptotically periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty} (f(t + \omega) - f(t)) = 0$. In this case, we say that ω is an asymptotic period of f and that f is S-asymptotically ω-periodic.