Nonlinear Analysis 71 (2009) 2447–2455

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

A variational approach to a quasilinear elliptic problem involving the p-Laplacian and nonlinear boundary condition

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ARTICLE INFO

Article history:
Received 2 January 2008
Accepted 15 January 2009

MSC:
35J50
35J55
35J65

Keywords:
Quasilinear elliptic problem
p-Laplacian
Nehari manifold
Nonlinear boundary condition

ABSTRACT

Using the technique of Brown and Wu [K.J. Brown, T.F. Wu, A semilinear elliptic system involving nonlinear boundary condition and sign changing weight function, J. Math. Anal. Appl. 337 (2008) 1326–1336], we present a note on the paper [T.F. Wu, A semilinear elliptic problem involving nonlinear boundary condition and sign-changing potential, Electron. J. Differential Equations 131 (2006) 1–15] by Wu. Indeed, we extend the multiplicity results for a class of semilinear problems to the quasilinear elliptic problems of the form:

$$\begin{cases} -\Delta_p u + m(x)|u|^{p-2}u = \lambda a(x)|u|^{q-2}u, & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial n} = b(x)|u|^{r-2}u, & x \in \partial\Omega. \end{cases}$$

Here Δ_p denotes the p-Laplacian operator defined by $\Delta_p z = \operatorname{div}(|\nabla z|^{p-2}\nabla z)$, $1 < q < p < r < p*$ ($p* = \frac{pN}{N-p}$ if $N > p$, $p* = \infty$ if $N \le p$), $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\frac{\partial}{\partial n}$ is the outer normal derivative, $\lambda \in \mathbb{R} \setminus \{0\}$, the weight $m(x)$ is a bounded function with $\|m\|_\infty > 0$ and $a(x), b(x)$ are continuous functions which change sign in $\overline{\Omega}$.

1. Introduction

The aim of this paper is to prove some existence and multiplicity results of nontrivial nonnegative solutions to the quasilinear elliptic problems:

$$\begin{cases} -\Delta_p u + m(x)|u|^{p-2}u = \lambda\, a(x)|u|^{q-2}u, & x \in \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = b(x)|u|^{r-2}u, & x \in \partial\Omega, \end{cases} \tag{1}$$

where Δ_p is the so-called p-Laplacian operator – i.e. $\Delta_p z = \operatorname{div}(|\nabla z|^{p-2}\nabla z)$, $1 < q < p < r < p*$ ($p* = \frac{pN}{N-p}$ if $N > p$, $p* = \infty$ if $N \le p$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $\frac{\partial}{\partial n}$ is the outer normal derivative, $\lambda \in \mathbb{R} \setminus \{0\}$, the weight $m(x)$ is a bounded function with $\|m\|_\infty > 0$ and a, b satisfy the following assumptions:

(H1) $a(x) \in C(\bar{\Omega})$ with $\|a\|_\infty = 1$;
(H2) $b(x) \in C(\partial\Omega)$ with $\|b\|_\infty = 1$.

Elliptic equations involving the p-Laplace operator arise in some physical models like the flow of non-Newtonian fluids: pseudo-plastic fluids correspond to $p \in (1, 2)$ while dilatant fluids correspond to $p > 2$. The case $p = 2$ expresses Newtonian

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0362-546X/$ – see front matter
doi:10.1016/j.na.2009.01.090

fluids [1]. On the other hand, quasilinear elliptic problems like (1) appear naturally in several branches of pure and applied mathematics, such as the study of optimal constants for the Sobolev trace embedding (see [2–5]); the theory of quasiregular and quasiconformal mappings in Riemannian manifolds with boundary (see [6,7]); non-Newtonian fluids, reaction diffusion problems, flow through porous media, nonlinear elasticity, glaciology, etc. (see [8,1,9,10]).

The motivation for our investigation is the case $p = 2$ (linear operator) and $m \equiv 1$ (no weight), that was studied by Wu [11]. In the recent paper, Brown and Wu [12] studied the multiplicity results of nontrivial nonnegative solutions for a semilinear elliptic system. It is our purpose in this paper to study the general situation of a nonlinear operator with weight, i.e., $1 < p < p^*$ and m indefinite in (1). Indeed, motivated by [11], and using recent ideas from [12], we shall establish the existence and multiplicity results for problem (1). In the case when $p = 2$, similar problems (with Dirichlet or Neumann boundary condition) have been studied by Drabek et al. [13,14], Ambrosetti–Brezis–Cerami [15] using variational methods, and by Amman and Lopez-Gomez [16] by using global bifurcation theory.

In recent years, several authors have used the Nehari manifold to solve semilinear and quasilinear problems (see [17–19, 12,20,20–22]). Brown and Zhang [23] have studied a subcritical semilinear elliptic equation with a sign-changing weight function and a bifurcation real parameter in the case $p = 2$ and Dirichlet boundary conditions. Exploiting the relationship between the Nehari manifold and fibering maps (i.e., maps of the form $t \longmapsto J_\lambda(tu)$ where J_λ is the Euler function associated with the equation), they gave an interesting explanation of the well-known bifurcation result. In fact, the nature of the Nehari manifold changes as the parameter λ crosses the bifurcation value. In this work, we give a variational method which is similar to the fibering method (see [24] or [18,23]) to prove the existence of at least two nontrivial nonnegative solutions of problem (1). In particular, by using the method of [12], we do this without the extraction of the Plais–Smale sequences in the Nehari manifold, as in [17,11].

This paper is divided into three sections, organized as follows. In Section 2, we give some notation, preliminaries, properties of the Nehari manifold and set up the variational framework of the problem. In Section 3, we give our main result.

2. Variational setting

Let $W_0^{1,s} = W_0^{1,s}(\Omega)$, $s > 1$, denote the usual Sobolev space. In the Banach space $W_0^{1,p}(\Omega) = W$ we introduce the norm

$$\|u\|_W = \left(\int_\Omega (|\nabla u|^p + m(x)|u|^p)\mathrm{d}x\right)^{\frac{1}{p}}$$

which is equivalent to the standard one. Throughout this paper, we set C and $\overline{C}$ as the best Sobolev and the best Sobolev trace constants for the embedding of $W_0^{1,p}(\Omega)$ in $L^q(\Omega)$ and $W_0^{1,p}(\Omega)$ in $L^r(\partial\Omega)$, respectively. First we give the definition of the weak solution of (1).

Definition 2.1. We say that $u \in W$ is a weak solution to (1) if for any $v \in W$ with $v \geq 0$ we have

$$\int_\Omega (|\nabla u|^{p-2}\nabla u.\nabla v + m(x)|u|^{p-2}u\, v\,)\mathrm{d}x = \lambda\int_\Omega a(x)|u|^{q-2}u\, v\,\mathrm{d}x + \int_{\partial\Omega} b(x)|u|^{r-2}u\, v\,\mathrm{d}s.$$

It is clear that problem (1) has a variational structure. Let $I_\lambda : W \to R$ be the corresponding energy functional of problem (1) which is defined by

$$I_\lambda(u) = \frac{1}{p}M(u) - \frac{\lambda}{q}A(u) - \frac{1}{r}B(u),$$

where

$$M(u) = \int_\Omega (|\nabla u|^p + m(x)|u|^p)\mathrm{d}x, \qquad A(u) = \int_\Omega a(x)|u|^q\,\mathrm{d}x, \quad \text{and} \quad B(u) = \int_{\partial\Omega} b(x)|u|^r\,\mathrm{d}s.$$

It is well known the weak solutions of Eq. (1) are the critical points of the energy functional I_λ. Let J be the energy functional associated with an elliptic problem on a Banach space X. If J is bounded below and J has a minimizer on X, then this minimizer is a critical point of J. So, it is a solution of the corresponding elliptic problem. However, the energy functional I_λ, is not bounded below on the whole space W, but is bounded on an appropriate subset, and a minimizer on this set (if it exists) gives rise to solution to (1).

Consider the Nehari minimization problem for $\lambda \in \mathbb{R} \setminus \{0\}$,

$$\alpha_\lambda = \inf\{I_\lambda(u) \,:\, u \in \mathcal{N}_\lambda\},$$

where $\mathcal{N}_\lambda = \{u \in W \setminus \{0\} : \langle I'_\lambda(u), u\rangle = 0\}$. It is easy to see that $u \in \mathcal{N}_\lambda$ if and only if

$$M(u) - \lambda A(u) = B(u). \tag{2}$$

Note that $\mathcal{N}_\lambda$ contains every nonzero solution of problem (1).