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Multiplicity of positive solutions for singular three-point boundary value problems at resonance

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ABSTRACT

In this paper, we study the existence and multiplicity of positive solutions for the three-point boundary value problems with singular nonlinear perturbations at resonance. The existence results are applicable to both the case of a strong singularity and the case of a weak singularity. The proof is based on a nonlinear alternative Leray–Schauder principle and a fixed point theorem in cones for completely continuous operators.

1. Introduction

In this paper, we consider the existence and multiplicity of positive solutions for the three-point boundary value problems

$$x'' + m^2 x = f(t,x) + e(t), \quad 0 < t < 1, \tag{1.1}$$

$$x'(0) = 0, \qquad x(\eta) = x(1), \tag{1.2}$$

where $m \in (0, \pi/2)$ is a positive constant and $\eta \in (0, 1)$ is given. The nonlinearity $f(t, x)$ is continuous and is assumed to have a singularity at $x = 0$. The problem (1.1)–(1.2) happens to be at resonance in the sense that the associated linear homogeneous boundary value problem

$$\begin{aligned} &x''(t) = 0, \quad 0 < t < 1, \\ &x'(0) = 0, \quad x(\eta) = x(1), \end{aligned} \tag{1.3}$$

has $x(t) = c$, $c \in \mathbb{R}$, as a nontrivial solution.

The study of multi-point boundary value problems for ordinary differential equations, which arise in many areas of applied mathematics and physics such as elastic stability and heat flow problems [1], goes back at least to Il'in and Moiseev [2] and, later, Gupta [3]. Since then, more general nonlinear multi-point boundary value problems have been studied by

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several authors; see [4–8]. They have attracted even more attention recently. The classical tools for such problems include the coincidence degree theory of Mawhin [9], the Leray–Schauder continuation theorem [9], the fixed point index theorem [8], the method of lower and upper solutions [10]. However, as far as the positive solutions are concerned, most results are for the non-resonance case [5,3,7]. For the existence of positive solutions of multi-point boundary value problems at resonance, there are only few related works; one can see [11,12].

On the other hand, the existence and multiplicity of positive problems for the singular equations have also been investigated extensively in the literature [13–20,25]. The singular problems appear in many fields and have many applications such as in nonlinear elasticity [16] and the Brillouin focusing system [21]. Eq. (1.1) is called singular at 0, which means that $f(t,x)$ becomes unbounded when $x \to 0^+$. Mathematically, Eq. (1.1) is a singular perturbation of a second-order equation. To our knowledge, research on the existence and multiplicity of positive solutions for multi-point problems concerning the singular nonlinearity at resonance is quite rarely seen [10].

In this paper, we will apply the nonlinear alternative Leray–Schauder principle and the fixed point theorem in cones to study multi-point problems (1.1)–(1.2) with a singular nonlinearity at resonance. We are concerned with both the case $\gamma_* \geq 0$ and the case $\gamma^* < 0$ (Section 2 for notation), which play an important role in the literature. The main aim is to get the existence results for the following singular problems (arising from chemical reactor theory [14] and nonlinear elasticity [16]):

$$x'' + m^2 x = b(t)x^{-\alpha} + \mu c(t)x^{\beta} + e(t), \quad 0 \leq t \leq 1, \tag{1.4}$$

with $b(t), c(t), e(t) \in \mathbb{C}[0,1]$, $\alpha, \beta > 0$ and $\mu \in \mathbb{R}$ a given parameter. Here, the Green function is not positive but non-negative. Meanwhile, we emphasize that $e(t)$ does not need to be positive, and we do not need the strong force condition $\alpha \geq 1$, which has been a standard condition in the literature for studying the singular problems [22].

The remaining part of the paper is organized as follows. In Section 2, some preliminary results will be given. In Section 3, the main results are presented and proved. Meanwhile, some applications are given.

Finally let us fix some notation to be used in the following: 'a.e.' means 'almost every' and 'for a.e.' means 'for almost every'. Given $a \in L^1[0,1]$, we write $a \succ 0$ if $a \geq 0$ for a.e. $t \in [0,1]$ and it is positive in a set of positive measure. For any $a \in \mathbb{C}[0,1]$, we will use the following notation: $a_* = \min_t a(t)$, $a^* = \max_t a(t)$.

2. Preliminaries

In this section, we present some preliminary results which will be needed in the following section.

Lemma 2.1 ([11]). *Suppose that $e : [0,1] \to [0,\infty)$ is continuous. Then the following problem:*

$$\begin{cases} x'' + m^2 x = e(t), & 0 < t < 1, \\ x'(0) = 0, & x(\eta) = x(1), \end{cases} \tag{2.1}$$

has a unique solution $x \in \mathbb{C}^2[0,1]$ with the representation

$$x = (\mathcal{L}e)(t) = \int_0^1 G(t,s)e(s)\mathrm{d}s, \tag{2.2}$$

where

$$G(t,s) = \begin{cases} \dfrac{\sin m(t-s)}{m}, & 0 \leq s \leq t \leq 1, \\ 0, & 0 \leq t \leq s \leq 1, \end{cases} + \frac{\cos mt}{\beta \sin \frac{m(\eta+1)}{2}} \begin{cases} \cos \dfrac{m(2s-\eta-1)}{2}, & 0 \leq s \leq \eta < 1, \\ \dfrac{\sin m(1-s)}{2\sin \frac{m(1-\eta)}{2}}, & 0 < \eta \leq s \leq 1, \end{cases} \tag{2.3}$$

is the Green function.

In particular, as $x(t) \equiv 1/m^2$ is the solution to (2.1) with $e(t) \equiv 1$, we have

$$(\mathcal{L}1)(t) = \int_0^1 G(t,s)\mathrm{d}s = 1/m^2 \quad \text{for all } t \in [0,1].$$

Lemma 2.2 ([11]). *There exists a continuous function $\phi : [0,1] \to [0,\infty)$ and a constant $\tau \in (0,1]$ such that*

$$\tau\phi(s) \leq G(t,s) \leq \phi(s), \quad t,s \in [0,1].$$

In fact, $\phi(s) = 1 - s$ is available.

Let $(X, \|\cdot\|)$ be a Banach space, we say that $K \in X$ is a cone if it is closed, nonempty, $K \neq \{0\}$ and whenever $x, y \in K$ and $\lambda, \mu \in \mathbb{R}$ with $\lambda > 0$, $\mu > 0$ then $\lambda x + \mu y \subset K$. If D is a subset of X, we write $D_K = D \cap K$ and $\partial_K D = (\partial D) \cap K$.

As usual, we define a compact map $T : X \to X$ as a continuous map such that $\overline{T(X)}$ is a compact subset of X. A map $T : X \to X$ is said to be completely continuous if it is continuous and $\overline{T(C)}$ is a compact subset of X for each bounded subset $C \subset X$.