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Random attractors for the stochastic FitzHugh–Nagumo system on unbounded domains

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ABSTRACT

The existence of a random attractor for the stochastic FitzHugh–Nagumo system defined on an unbounded domain is established. The pullback asymptotic compactness of the stochastic system is proved by uniform estimates on solutions for large space and time variables. These estimates are obtained by a cut-off technique.

1. Introduction

In this paper, we investigate the asymptotic behavior of solutions of the following stochastic FitzHugh–Nagumo system defined on $\mathbb{R}^n$:

$$\begin{aligned} &\mathrm{d}u + (\lambda u - \Delta u + \alpha v)\mathrm{d}t = (f(x,u) + g)\mathrm{d}t + \phi_1 \mathrm{d}w_1, \quad x \in \mathbb{R}^n,\ t > 0, \\ &\mathrm{d}v + (\delta v - \beta u)\mathrm{d}t = h\mathrm{d}t + \phi_2 \mathrm{d}w_2, \quad x \in \mathbb{R}^n,\ t > 0, \end{aligned}$$

where λ, α, δ and β are positive constants, $g \in L^2(\mathbb{R}^n)$ and $h \in H^1(\mathbb{R}^n)$ are given, $\phi_1 \in H^2(\mathbb{R}^n) \cap W^{2,p}(\mathbb{R}^n)$ for some $p \geq 2$, $\phi_2 \in H^1(\mathbb{R}^n)$, f is a nonlinear function satisfying certain dissipative conditions, w_1 and w_2 are independent two-sided real-valued Wiener processes on a probability space which will be specified later.

The concept of random attractor was introduced in [1,2], which is an analogue of global attractors for deterministic dynamical systems as studied in [3–7]. When PDEs are defined in *bounded* domains, the existence of random attractors has been investigated by many authors, see, e.g., [8–10,1,2] and the references therein. However, this problem is not well studied in the case of *unbounded* domains. Recently, the existence of random attractors for some PDEs defined on unbounded domains was established in [11–13]. For the stochastic Navier–Stokes equations on unbounded domains, the asymptotic compactness and existence of bounded absorbing sets were proved in [14]. In this paper, we study random attractors for the stochastic FitzHugh–Nagumo system defined on $\mathbb{R}^n$.

Notice that Sobolev embeddings are not compact when domains are unbounded. This introduces a major obstacle for proving existence of attractors for PDEs on unbounded domains. For some deterministic PDEs, such difficulty can be

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overcome by the energy equation approach that was developed by Ball in [15,16] and used in [17–23]. Under certain circumstances, the tail-estimates method must be used to deal with the problem caused by the unboundedness of domains. This approach was developed in [24] (see also [25–32]) for deterministic PDEs, and used in [11–13] for stochastic systems. In this paper, we will use the idea of uniform estimates on the tails of solutions to investigate the asymptotic behavior of the stochastic FitzHugh–Nagumo system.

This paper is organized as follows. In the next section, we review the pullback random attractors theory for random dynamical systems. In Section 3, we define a continuous random dynamical system for the stochastic FitzHugh–Nagumo system on $\mathbb{R}^n$. Then we derive the uniform estimates of solutions in Section 4, which include the uniform estimates on the tails of solutions. Finally, we establish the asymptotic compactness of the random dynamical system and prove the existence of a pullback random attractor.

In what follows, we adopt the following notations. We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the inner product of $L^2(\mathbb{R}^n)$, respectively. The norm of a given Banach space X is written as $\|\cdot\|_X$. We also use $\|\cdot\|_p$ to denote the norm of $L^p(\mathbb{R}^n)$. The letters c and c_i $(i = 1, 2, \ldots)$ are generic positive constants which may change their values from line to line or even in the same line.

2. Preliminaries

In this section, we recall some basic concepts related to random attractors for stochastic dynamical systems. The reader is referred to [8,33,10,2] for more details.

Let $(X, \|\cdot\|_X)$ be a separable Hilbert space with Borel σ-algebra $\mathcal{B}(X)$, and $(\Omega, \mathcal{F}, P)$ be a probability space.

Definition 2.1. $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is called a metric dynamical system if $\theta : \mathbb{R} \times \Omega \to \Omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable, θ_0 is the identity on Ω, $\theta_{s+t} = \theta_t \circ \theta_s$ for all $s, t \in \mathbb{R}$ and $\theta_t P = P$ for all $t \in \mathbb{R}$.

Definition 2.2. A continuous random dynamical system (RDS) on X over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$ is a mapping

$$\Phi : \mathbb{R}^+ \times \Omega \times X \to X \quad (t, \omega, x) \mapsto \Phi(t, \omega, x),$$

which is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable and satisfies, for P-a.e. $\omega \in \Omega$,

(i) $\Phi(0, \omega, \cdot)$ is the identity on X;
(ii) $\Phi(t + s, \omega, \cdot) = \Phi(t, \theta_s\omega, \cdot) \circ \Phi(s, \omega, \cdot)$ for all $t, s \in \mathbb{R}^+$;
(iii) $\Phi(t, \omega, \cdot) : X \to X$ is continuous for all $t \in \mathbb{R}^+$.

Hereafter, we always assume that Φ is a continuous RDS on X over $(\Omega, \mathcal{F}, P, (\theta_t)_{t\in\mathbb{R}})$.

Definition 2.3. A random bounded set $\{B(\omega)\}_{\omega\in\Omega}$ of X is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if for P-a.e. $\omega \in \Omega$,

$$\lim_{t\to\infty} e^{-\beta t} d(B(\theta_{-t}\omega)) = 0 \quad \text{for all } \beta > 0,$$

where $d(B) = \sup_{x\in B} \|x\|_X$.

Definition 2.4. Let $\mathcal{D}$ be a collection of random subsets of X. Then $\mathcal{D}$ is called inclusion-closed if $D = \{D(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and $\tilde{D} = \{\tilde{D}(\omega) \subseteq X : \omega \in \Omega\}$ with $\tilde{D}(\omega) \subseteq D(\omega)$ for all $\omega \in \Omega$ imply that $\tilde{D} \in \mathcal{D}$.

Definition 2.5. Let $\mathcal{D}$ be a collection of random subsets of X and $\{K(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then $\{K(\omega)\}_{\omega\in\Omega}$ is called an absorbing set of Φ in $\mathcal{D}$ if for every $B \in \mathcal{D}$ and P-a.e. $\omega \in \Omega$, there exists $t_B(\omega) > 0$ such that

$$\Phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq K(\omega) \quad \text{for all } t \geq t_B(\omega).$$

Definition 2.6. Let $\mathcal{D}$ be a collection of random subsets of X. Then Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for P-a.e. $\omega \in \Omega$, $\{\Phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty}$ has a convergent subsequence in X whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$.

Definition 2.7. Let $\mathcal{D}$ be a collection of random subsets of X and $\{\mathcal{A}(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$. Then $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is called a $\mathcal{D}$-random attractor (or $\mathcal{D}$-pullback attractor) for Φ if the following conditions are satisfied, for P-a.e. $\omega \in \Omega$,

(i) $\mathcal{A}(\omega)$ is compact, and $\omega \mapsto d(x, \mathcal{A}(\omega))$ is measurable for every $x \in X$;
(ii) $\{\mathcal{A}(\omega)\}_{\omega\in\Omega}$ is invariant, that is,

$$\Phi(t, \omega, \mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega), \quad \forall\, t \geq 0;$$