Nonlinear Analysis 71 (2009) 2927–2935

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Non-self mappings satisfying non-linear contractive condition with applications

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ARTICLE INFO

Article history:
Received 21 January 2009
Accepted 21 January 2009

Keywords:
Metric space of hyperbolic type
Coincidence point
Common fixed point
Coincidentally commuting mapping
Complete metric space

ABSTRACT

The concept of non-self mappings satisfying a new non-linear contractive type condition is introduced, and coincidence and common fixed point theorems in metric spaces of hyperbolic type are proved. Presented theorems generalize and improve recent theorems of Imdad and Kumar [M. Imdad, S. Kumar, Rhoades-type fixed-point theorems for a pair of nonself mappings, Comput. Math. Appl. 46 (2003), 919–927] and several others theorems. One example is constructed to show that our theorems are genuine generalizations of the above mentioned results of Imdad and Kumar and several others.

1. Introduction

The theory of contractive self-mappings have been developed in many directions (cf. [1–7] and references therein). Many fixed point theorems for contraction self-mappings have found applications in diverse disciplines of mathematics, engineering and economics. In convex spaces there are cases where the involved function is not necessarily a self-mapping of a closed subset. Assad [8] and Assad and Kirk [9] first studied non-self contraction mappings in a metric space (X, d), metrically convex in the sense of Menger (that is, for each x, y in X with $x \neq y$ there exists z in X, $x \neq z \neq y$, such that $d(x, z) + d(z, y) = d(x, y)$). In recent years, this technique has been developed and fixed and common fixed-points of non-self mappings have been studied by many authors (cf. [8,10–19]). Some of obtained results have find applications (cf. [9, 18–20]). In numerical mathematics especially the restricted condition $T(\partial K) \subseteq K$ is preferred instead of $T(K) \subseteq K$; where K is a closed subset of X, $T : K \to X$ and ∂K is the boundary of K.

In an attempt to generalize a theorem of Assad [8] and Assad and Kirk [9], Rhoades [18] proved the following result in a Banach space.

Theorem 1. *Let X be a Banach space, K a nonempty closed subset of X and $T : K \to X$ a mapping of K into X satisfying the condition*

$$d(Tx, Ty) \leq h \max\left\{\frac{d(x, y)}{2}, d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{q}\right\} \tag{1}$$

for all x, y in K, $0 < h < 1$, $q \geq 1 + 2h$ and T has the additional property that for each $x \in \partial K$, the boundary of K, $Tx \in K$, then T has a unique fixed point.

Ćirić in [10] obtained a slightly extension of Theorem of Rhoades [18]. Recently Imdad and Kumar [15] generalized the results of Rhoades [18] and Ćirić [10]. They proved the following theorem.

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0362-546X/$ – see front matter
doi:10.1016/j.na.2009.01.174

Theorem 2. *Let X be a Banach space, K a nonempty closed subset of X and $F, T : K \to X$ two mappings satisfying the condition*

$$d(Fx, Fy) \le h \max\left\{\frac{d(Tx, Ty)}{2}, d(Tx, Fx), d(Ty, Fy), \frac{d(Tx, Fy) + d(Ty, Fx)}{q}\right\} \tag{2}$$

for all x, y in K, $0 < h < 1$, $q \ge 1 + 2h$ and

(i) $\partial K \subseteq TK$, $FK \cap K \subset TK$,
(ii) $Tx \in \partial K \Longrightarrow Fx \in K$, *and*
(iii) TK *is closed in* X.

Then there exists a coincidence point z in X. Moreover, if F and T are coincidentally commuting, then z remains a unique common fixed point of F and T.

Observe that if F and T satisfy (2), then F and T also satisfy the following condition:

$$d(Fx, Fy) \le h_1 \max\left\{\frac{d(Tx, Ty)}{2}, d(Tx, Fx), d(Ty, Fy), \frac{d(Tx, Fy) + d(Ty, Fx)}{3}\right\}, \tag{3}$$

where $h_1 = 3h/(1 + 2h) < 1$.

Recall (see Jungck and Rhoades [21]) that a pair of mappings (F, T), defined on a nonempty set S, is said to be coincidentally commuting, if $Tx = Fx \Longrightarrow FTx = TFx$; $x \in S$.

The purpose of this paper is to introduce the concept of non-self mappings which satisfy a new non-linear contractive type condition, weaker than (2), and to prove common fixed point theorems in metric spaces of hyperbolic type. Our theorems generalize the main theorems of Assad [8], Ćirić [10], Khan et al. [17], Rhoades [18] and Imdad and Kumar [15] in many aspects. One example is constructed to show that our results are genuine generalizations of the known results from this area.

2. Main results

Let (X, d) be a convex metric space which contains a family L of metric segments (isometric images of real line segments) such that

(a) each two points x, y in X are end points of exactly one member seg$[x, y]$ of L, and
(b) if u, x, y in X and if $z \in \text{seg}[x, y]$ satisfies $d(x, z) = \lambda d(x, y)$ for any $\lambda \in [0, 1]$, then

$$d(u, z) \le (1 - \lambda)d(u, x) + \lambda d(u, y). \tag{4}$$

A space of this type is said to be *a metric space of hyperbolic type* (Takahashi [22] uses the term*a convex metric space*). This class includes all normed linear spaces, as well as all spaces with hyperbolic metric (see [23] for a discussion). For instance, if X is a Banach space, then

$$\text{seg}[x, y] = \{(1 - \lambda)x + \lambda y : 0 \le \lambda \le 1\}.$$

A linear space with a translation invariant metric satisfying

$$d(\lambda x + (1 - \lambda)y, 0) \le \lambda d(x, 0) + (1 - \lambda)d(y, 0)$$

is also a metric space of hyperbolic type. There are many other examples but we consider these as paradigmatic.

Now we shall prove a common fixed point theorem for a pair of non-self mappings satisfying a new non-linear contraction type condition in a metric space of hyperbolic type. Recall that the concept of non-linear contractions was introduced and studied by Boyd and Wong [2], and that some applications of non-linear contractions was considered by Camargo [24], Tsachev and Angelov [19] and Zeng et al. [20].

Theorem 3. *Let X be a metric space of hyperbolic type, K a non-empty closed subset of X and ∂K the boundary of K. Let ∂K be non-empty and let $T : K \to X$ and $F : K \cap T(K) \to X$ be two non-self mappings such that for every $x, y \in X$, there exists*

$$u \in C(x, y) \equiv \left\{d(Tx, Fx), d(Ty, Fy), \min\{d(Tx, Fy), d(Ty, Fx)\}, \frac{d(Tx, Ty)}{2}, \frac{\max\{d(Tx, Fy), d(Ty, Fx)\}}{2}\right\},$$

such that

$$d(Fx, Fy) \le \varphi(u), \tag{5}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing real function satisfying $\varphi(t+) < t$ for all $t > 0$.

Suppose that F and T have the additional properties:

(i) $\partial K \subseteq T(K)$;
(ii) $F(K) \cap K \subseteq T(K)$;