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Existence of positive entire solutions of a semilinear elliptic problem with a gradient term

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ABSTRACT

By a sub–supersolution argument and a perturbed argument, we show the existence of entire solutions to a semilinear elliptic problem $-\Delta u + h(x)|\nabla u|^q = b(x)g(u)$, $u > 0$, $x \in \mathbb{R}^N$, $\lim_{|x|\to\infty} u(x) = 0$, where $q \in (1, 2]$, $b, h \in C^{\alpha}_{\text{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$, $h(x) \geq 0$, $b(x) > 0$, $\forall\, x \in \mathbb{R}^N$, and $g \in C^1((0, \infty), (0, \infty))$ which may be singular at 0. No monotonicity condition is imposed on the functions $g(s)$ and $g(s)/s$.

1. Introduction and the main results

The purpose of this note is to investigate the existence of entire solutions to the following model problem

$$-\Delta u + h(x)|\nabla u|^q = b(x)g(u), \quad u > 0,\ x \in \mathbb{R}^N,\ \lim_{|x|\to\infty} u(x) = 0, \tag{1.1}$$

where $q \in (1, 2]$, $h \in C^{\alpha}_{\text{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$ is non-negative in Ω, g satisfies

(g_1) $g \in C^1((0, \infty), (0, \infty))$;
(g_2) $\lim_{s\to 0^+} g(s)/s = \infty$;
(g_3) $\lim_{s\to\infty} g(s)/s = 0$;

and b satisfies

(b_1) $b \in C^{\alpha}_{\text{loc}}(\mathbb{R}^N)$ and $b(x) > 0,\ \forall\, x \in \mathbb{R}^N$;
(b_2) the linear problem

$$-\Delta u = b(x), \quad u > 0,\ x \in \mathbb{R}^N,\ \lim_{|x|\to\infty} u(x) = 0 \tag{1.2}$$

has a unique solution $w \in C^{2+\alpha}_{\text{loc}}(\mathbb{R}^N)$.

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First, let us review the following model

$$-\Delta u = b(x)g(u), \quad u > 0,\ x \in \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0. \tag{1.3}$$

Problem (1.3) arises from many branches of mathematics and applied mathematics. It was discussed and extended to more general problems in a number of works, for instance, [1–13]. When the equation is considered over a bounded smooth domain Ω instead of $\mathbb{R}^N$, the corresponding problem was studied, for example, in [14–25] and the references cited therein.

For $g(u) = u^{-\gamma}$ with $\gamma > 0$, if b satisfies (b_1) and the following condition

(b_3) $\int_0^\infty r\phi(r)\mathrm{d}r < \infty$, where $\phi(r) = \max_{|x|=r} b(x)$,

Lair and Shaker [9] showed that problem (1.3) has a unique solution $u \in C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. Later, Lair and Shaker [10] and Zhang [13] extended the above result to the more general g which satisfies (g_1) and

(g_4) g is non-increasing on $(0, \infty)$ and $\lim_{s\to 0^+} g(s) = \infty$.

Cîrstea and Rădulescu [3] also extended the above results to the more general g which satisfies (g_1), (g_2) and

(g_5) $\frac{g(s)}{s+s_0}$ is decreasing on $(0, \infty)$ for some $s_0 > 0$;
(g_6) g is bounded in a neighborhood of ∞.

Recently, Dinu [26] further generalized the above results to the cases that

(i) b satisfies (b_1) and (b_3);
(ii) g satisfies (g_1), (g_3) and
(g_7) $\frac{g(s)}{s}$ is decreasing on $(0, \infty)$;

and $\lim_{|x|\to\infty} u(x) = l > 0$ instead of $\lim_{|x|\to\infty} u(x) = 0$ in problem (1.3); or the following cases that

(i_1) b satisfies (b_1) and
(b_4) $\int_0^\infty r^{N-1}\phi(r)\mathrm{d}r < \infty$, where $N \geq 3$;
(ii_2) g satisfies (g_1)–(g_3) and
(g_8) g is increasing on $(0, \infty)$.

Afterwards, Goncalves and Santos [8] also generalized the above results to the case that g satisfies (g_1)–(g_3) and (g_7). Ye and Zhou [12, Theorem 4.2] showed that if g satisfies (g_1) and is non-increasing on $(0, \infty)$, b satisfies (b_1), then problem (1.3) admits a solution if and only if b satisfies (b_2). Moreover, if a solution of problem (1.3) exists, it is unique.

Now let us return to problem (1.1).

When $g(u) = u^{-\gamma}$ with $\gamma > 0$, $q \in (1, 2]$, $h \in C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ is non-negative in $\mathbb{R}^N$, b satisfies (b_1) and (b_3), Dinu [26] showed that problem (1.1) has a unique solution $u \in C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. Recently, the author [27] showed that when$q \in (0, 1)$, $h \in C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$, $h(x) < 0$, $\forall\, x \in \mathbb{R}^N$, g satisfies (g_1)–(g_3), and b satisfies (b_3) instead of (b_2), problem (1.1) have at least one solution.

In this paper we continue to consider the existence of entire solutions to problem (1.1) for the functions $g(s)$ and $g(s)/s$ which do not have monotonicity.

Our main result is summarized in the following theorem.

Theorem 1.1. *Let $q \in (1, 2]$, $h \in C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ be non-negative in $\mathbb{R}^N$, and b satisfy (b_1) and (b_2). If g satisfies (g_1)–(g_3), then problem (1.1) has at least one solution $u \in C^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$.*

Remark 1.1 ([27]). The condition (b_3) implies (b_2), but (b_2) is invariant under translations.

Remark 1.2. Some basic examples of the functions, which satisfy (g_1)–(g_3), are

(i) $u^{-\gamma} + u^p + \sin f(u) + 1$, where $\gamma > 0$, $p < 1$ and $f \in C^2(\mathbb{R})$;
(ii) $\mathrm{e}^{1/u^\gamma} + u^p + \cos f(u) + 1$, where $\gamma > 0$, $p < 1$ and $f \in C^2(\mathbb{R})$;
(iii) $u^{-\gamma}\ln^{-q_1}(1+u) + \ln^{q_2}(1+u) + u^p + \sin f(u) + 2$ with $f \in C^2(\mathbb{R})$, $\gamma > 0$, $p < 1$, $q_2 > 0$ and $q_1 > 0$;
(iv) $u^{-\gamma} + \arctan f(u) + \pi$ with $f \in C^2(\mathbb{R})$ and $\gamma > 0$.

Remark 1.3. The technique of this paper in our proofs can be applied to the more general problem

$$-\Delta u + h(x)|\nabla u|^q = b(x)g(u) + a(x)f(u), \quad u > 0,\ x \in \mathbb{R}^N, \quad \lim_{|x|\to\infty} u(x) = 0,$$

where $q \in (1, 2]$, $h \in C^{\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ is non-negative in $\mathbb{R}^N$, a and b satisfy (b_1) and (b_2), g and f satisfy (g_1), $g+f$ satisfies (g_2)–(g_3).

The paper is organized as follows. In Section 2 we give some preliminary considerations. Finally we show the existence of solutions to problem (1.1).