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# Classification of the centers and isochronous centers for a class of quartic-like systems

Jaume Llibre [a], Clàudia Valls [b,*]

[a] Departamento de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

[b] Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais 1049-001, Lisboa, Portugal

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ABSTRACT

In this paper we classify the centers and isochronous centers for a class of polynomial differential systems in $\mathbb{R}^2$ of degree $d$ that in complex notation $z = x + iy$ can be written as

$$\dot{z} = iz + (z\bar{z})^{\frac{d-4}{2}}(Az^3\bar{z} + Bz^2\bar{z}^2 + C\bar{z}^4),$$

where $d \geq 4$ is an arbitrary even positive integer and $A, B, C \in \mathbb{C}$. Note that if $d = 4$ we obtain a special case of quartic polynomial differential systems which can have a center at the origin.



## 1. Introduction and statement of the main results

In this paper we consider the polynomial differential systems in the real $(x, y)$-plane that has a singular point at the origin with eigenvalues $\pm$i and that can be written as

$$\dot{z} = iz + (z\bar{z})^{\frac{d-4}{2}}(Az^3\bar{z} + Bz^2\bar{z}^2 + C\bar{z}^4), \tag{1}$$

where $z = x + iy$, $d \geq 4$ is an arbitrary even positive integer and $A, B, C \in \mathbb{C}$. The vector field associated with this system is formed by the linear part i$z$ and by a homogeneous polynomial of degree $d$ formed by three monomials in complex notation. In real notation the number of monomials increases with $d$. Clearly the origin is either a weak focus or a center, see for instance [1,2].

For such systems we want to determine the conditions that ensure that the origin of (1) is a center or an isochronous center. Of course, these systems for $d = 4$ coincide with a class of quartic polynomial differential system. So we call the class of polynomial differential system (1) of even degree $d \geq 4$ the *quartic-like systems*.

The problem of characterizing the centers and isochronous centers has attracted the attention of many authors. However there is only one family of polynomial differential systems for which we have the complete classification of the centers and isochronous centers, the family of quadratic polynomial differential systems, or simply quadratic systems. Quadratic systems having a center were classified by Dulac in [3], Kapteyn in [4,5], Bautin in [6], Żołądek in [7], and quadratic systems having an isochronous center where characterized by Loud in [8]. The centers of the cubic systems with homogeneous nonlinearities were classified by [9,10], and the isochronous centers for such cubic systems were characterized by Pleshkan in [11]. However we are very far away from obtaining a complete classification of the centers and of the isochronous centers for the polynomial differential systems of degree 3, and even more so if the degree is higher. For a good survey on the known results on isochronous centers see [12].

* Corresponding author.
*E-mail addresses:* jllibre@mat.uab.cat (J. Llibre), cvalls@math.ist.utl.pt (C. Valls).

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The main result in this paper is the following one.

**Theorem 1.** *For $d \geq 4$ even the following statements holds for system (1).*

(a) *It has a center at the origin if and only if one of the following two conditions holds.*
  (a.1) $3A + 2\overline{B} = 0$,
  (a.2) $\mathrm{Im}(AB) = \mathrm{Im}(A^5C) = \mathrm{Im}(\overline{B}^5C) = 0$.

(b) *It has an isochronous center at the origin if and only if one of the following two conditions holds.*
  (b.1) $C = 0, B = \overline{A}$,
  (b.2) $C = 0, B = (d-2)\overline{A}/d$.

The paper is devoted to the proof of Theorem 1. To do so we have structured the paper as follows. In Section 2 we introduce some preliminaries that will be used in the paper, in Section 3 we provide the proof of Theorem 1(a), while the proof of Theorem 1(b) is given in Section 4.

## 2. Preliminaries

The resolution of Theorem 1 needs the effective computation of the Liapunov constants as well as of the period constants. We write

$$A = a_1 + \mathrm{i}a_2, \qquad B = b_1 + \mathrm{i}b_2, \qquad C = c_1 + \mathrm{i}c_2.$$

Now in polar coordinates $(r, \theta)$, given by $r^2 = z\overline{z}$ and $\theta = \arctan(\mathrm{Im}z/\mathrm{Re}z)$, system (1) becomes

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{F(\theta)\, r^d}{1 + G(\theta)\, r^{d-1}}, \tag{2}$$

where

$$\begin{aligned} F(\theta) &= (a_1 + b_1)\cos\theta - (a_2 - b_2)\sin\theta + c_1\cos 5\theta + c_2 \sin 5\theta, \\ G(\theta) &= (a_2 + b_2)\cos\theta + (a_1 - b_1)\sin\theta + c_2\cos 5\theta - c_1 \sin 5\theta. \end{aligned} \tag{3}$$

If system (1) has a center for $r > 0$ sufficiently small all the periodic orbits of this center are contained in the region $\dot{\theta} > 0$, and consequently they are also periodic orbits of the differential equation (2).

The transformation $(r, \theta) \mapsto (\rho, \theta)$ defined by

$$\rho = \frac{r^{d-1}}{1 + G(\theta)\, r^{d-1}}, \tag{4}$$

in a sufficiently small neighborhood of the origin is a diffeomorphism with its image. As far as we know the first in use in this transformation was Cherkas in [13]. If we write Eq. (2) in the variable $\rho$, we obtain the following Abel differential equation

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}\theta} &= [-(d-1)G(\theta)F(\theta)]\rho^3 + [(d-1)F(\theta) - G'(\theta)]\rho^2 \\ &= A(\theta)\rho^3 + B(\theta)\rho^2 + C\rho. \end{aligned} \tag{5}$$

The solution $\rho(\theta, \gamma)$ of (5) satisfying that $\rho(0, \gamma) = \gamma$ can be expanded in a convergent power series of $\gamma \geq 0$ when $\gamma$ is sufficiently small. Thus

$$\rho(\theta, \gamma) = \rho_1(\theta)\gamma + \rho_2(\theta)\gamma^2 + \rho_3(\theta)\gamma^3 + \cdots \tag{6}$$

with $\rho_1(\theta) = 1$ and $\rho_k(0) = 0$ for $k \geq 2$. Let $P : [0, \gamma_0] \to \mathbb{R}$ be the Poincaré map defined by $P(\gamma) = \rho(2\pi, \gamma)$ and for a convenient $\gamma_0 > 0$. Then the values of $\rho_k(2\pi)$ for $k \geq 2$ control the behavior of the Poincaré map in a neighborhood of $\rho = 0$. Therefore system (1) has a center at the origin if and only if $\rho_1(2\pi) = 1$ and $\rho_k(2\pi) = 0$ for every $k \geq 2$. Assuming that $\rho_2(2\pi) = \cdots = \rho_{m-1}(2\pi) = 0$ we say that $v_m = \rho_m(2\pi)$ is the $m$-th *Liapunov* or *Liapunov–Abel* constant of system (1).

The set of coefficients for which all the Liapunov constants vanish is called the *center variety* of the family of polynomial differential systems (1). By the Hilbert Basis Theorem the center variety is an algebraic set. Necessary conditions to have a center at the origin will be obtained by finding the zeros of the Liapunov constants.

We note that the center manifold, i.e. the space of systems (1) with a center at the origin, is invariant with respect to the action group $C^*$ of changes of variables $z \to \xi z$:

$$A \to \xi^{(d-6)/2}\overline{\xi}^{(d-4)/2}\xi^3\overline{\xi}A, \qquad B \to \xi^{(d-6)/2}\overline{\xi}^{(d-4)/2}\xi^2\overline{\xi}^2B, \qquad C \to \xi^{(d-6)/2}\overline{\xi}^{(d-4)/2}\overline{\xi}^4C.$$

In order to show the sufficiency of the found conditions for the center manifold, we will use the so-called *Linear Type Center Theorem* by looking for the existence of a local analytic first integral defined in a neighborhood of the origin, or we will show that system (1) is *reversible*. We recall that system (1) is *reversible* with respect to a straight line if it is invariant under the change of variables $\overline{w} = \mathrm{e}^{\mathrm{i}\gamma}z$, $\tau = -t$ for some $\gamma$ is real. For system (1) we have the following result whose proof can be found in [14].

**Lemma 2.** *System (1) is reversible if and only if $A = -\overline{A}\mathrm{e}^{\mathrm{i}\gamma}$, $B = -\overline{B}\mathrm{e}^{-\mathrm{i}\gamma}$ and $C = -\overline{C}\mathrm{e}^{-5\mathrm{i}\gamma}$, for some $\gamma \in \mathbb{R}$. Furthermore in this situation the origin of system (1) is a center.*