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Similarity solutions for a multi-dimensional replicator dynamics equation

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ABSTRACT

We construct a one-parameter family of self-similar solutions for a nonlinear degenerate multi-dimensional parabolic equation containing a nonlocal term. All these solutions are strictly positive and their integral over the whole space is 1. The equation serves as a replicator dynamics model where the set of strategies is a continuum.
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1. Introduction

The *replicator dynamics models* are popular models in evolutionary game theory. They have significant applications in economics, population biology, as well as in other areas of science.

Replicator dynamics have been studied extensively in the finite-dimensional case:

Let $A = (a_{ij})$ be an $m \times m$ negative matrix. The typical replicator dynamics equation is

$$u_i'(t) = \left[\sum_{j=1}^{m} a_{ij} u_j(t) - \sum_{k=1}^{m} \sum_{j=1}^{m} a_{kj} u_k(t) u_j(t) \right] u_i(t), \quad t > 0, \; i = 1, \dots, m, \tag{1}$$

which can also be symbolically written in the form

$$u_t = [Au - (u, Au)]\, u$$

(where $(Au)u$ is the vector whose ith component is the product of the ith components of (Au) and u). The matrix A is called the *payoff matrix* while $S = \{1, \dots, m\}$ is the *strategy space* and the vector

$$u = (u_1(t), \dots, u_m(t))^\top,$$

is a probability distribution on S, hence we must have

$$u_j(t) \geq 0, \quad \text{for } j = 1, \dots, m, \quad \text{and} \quad \sum_{j=1}^{m} u_j(t) = 1. \tag{2}$$

It is easy to see that if the conditions (2) are satisfied for $t = 0$, then they are satisfied for all $t \geq 0$ (under the flow (1)).

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The term in the square brackets on the right-hand side of Eq. (1) is a measure of the success of strategy i and it is assumed to be the difference of the payoff of the players playing strategy i from the average payoff of the population. It is then assumed that the logarithmic derivative of $u_i(t)$, where u_i is the percentage of the population playing i, is equal to this success measure, i.e. agents update their strategies proportionally to the success of the strategy i. This model was introduced in [1,2] (see also *Wikipedia* or [3] where a stochastic version of the model is discussed).

Infinite-dimensional versions of these evolutionary strategy models have been proposed, e.g., in [4,5] (see also the companion paper [6]) in connection with certain economic examples. However, the abstract form of the proposed equations does not allow one to obtain insight into the form of solutions. In order to make some progress in this direction, in the recent work [7] the authors restricted their attention to the case where the strategy space S is the set $\mathbb{R}$ (i.e. the real line) and the payoff operator A is the differential operator $\mathrm{d}^2/\mathrm{d}x^2$. Then (1) becomes the evolution law

$$u_t = [u_{xx} - (u, u_{xx})]\, u, \tag{3}$$

where $(\cdot, \cdot)$ denotes the usual inner product in the Hilbert space $L_2(\mathbb{R})$ of the squared-integrable real-valued functions defined on $\mathbb{R}$. The initial condition, $u(x, 0)$ is taken to be the density of a probability measure on $\mathbb{R}$.

Eq. (3) is a nonlinear degenerate parabolic PDE with a nonlocal term. In [7] the authors constructed a one-parameter family of self-similar solutions for (3), namely solutions u of the form

$$u(t, x) = \frac{1}{t^\alpha} g\left(\frac{x}{t^\beta}\right).$$

All these similarity solutions are probability densities on $\mathbb{R}$, for every $t > 0$.

It is worth saying that there are situations where strategies correspond to geographical points and hence it is natural to model the set of strategies by a continuum. Also, the infinite-dimensional models lead to interesting mathematics (nonlinear, nonlocal, degenerate parabolic PDE's with rich structure).

In the present work we study the d-dimensional case (with $d \geq 2$) where the strategy space is $S = \mathbb{R}^d$, while $A = \Delta$, namely the Laplacian acting on $\mathbb{R}^d$. In this case the corresponding replicator dynamics problem takes the form

$$u_t = [\Delta u - (u, \Delta u)]\, u, \quad t > 0,\ x \in \mathbb{R}^d, \tag{4}$$

with

$$\int_{\mathbb{R}^d} u(0, x)\mathrm{d}x = 1 \quad \text{and} \quad u(0, x) \geq 0, \quad \text{for } x \in \mathbb{R}^d. \tag{5}$$

Here $(\cdot, \cdot)$ denotes the usual inner product on the Hilbert space $L_2(\mathbb{R}^d)$, i.e.

$$(f, g) = \int_{\mathbb{R}^d} f(x) g(x) \mathrm{d}x.$$

The main result of the article is the construction of a one-parameter family of self-similar solutions for (4) and (5), namely solutions u of the form

$$u(t, x) = \frac{1}{t^\alpha} g_d\left(\frac{r}{t^\beta}\right), \quad \text{where } r = |x| = \sqrt{x_1^2 + \cdots + x_d^2}. \tag{6}$$

All the solutions we obtained are probability densities on $\mathbb{R}^d$, for all $t > 0$. It is rather unusual for a parabolic problem to have an infinitude of such solutions, since they all approach $\delta(x)$, as $t \to 0^+$.

2. The Equation for g_d

Let $u(t, x)$ be a solution of (4). By applying integration by parts (i.e. the Divergence Theorem for the vector field $u\nabla u$—also known as Green's 1st identity) one obtains

$$(u, \Delta u) = \int_{\mathbb{R}^d} u \Delta u = -\int_{\mathbb{R}^d} \nabla u \cdot \nabla u = -\int_{\mathbb{R}^d} |\nabla u|^2,$$

provided that

$$\lim_{R\to\infty} \int_{S^{d-1}(R)} u \frac{\partial u}{\partial n} = 0, \tag{7}$$

where $S^{d-1}(R)$ is the sphere of radius R in $\mathbb{R}^d$, centered at the origin, and n its outward unit normal vector.

Thus, under (7), Eq. (4) can be written in the equivalent form

$$u_t = \left(\Delta u + \int_{\mathbb{R}^d} |\nabla u|^2\right) u, \quad t > 0,\ x \in \mathbb{R}^d. \tag{8}$$

Let us introduce the variable

$$s = t^{-\beta} r \tag{9}$$

(notice that $0 < s < \infty$). Then u of (6) can be also written as

$$u(t, x) = t^{-\alpha} g\,(s). \tag{10}$$

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